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Ion thermal effects in oscillating multi-ion plasma sheath theory

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(Received 10 October 2008; accepted 6 November 2008; published online 9 December 2008)

The effects of the ion temperature are discussed in a two-ion electron plasma and for a model applicable to the oscillating sheath theory that has recently been much of the focus in research. The differences between the fluid and kinetic models are pointed out, as well as the differences between the approximative kinetic description (which involves the expansion of the plasma dispersion function), and the exact kinetic description. It is shown that the approximative kinetic description, first, cannot describe the additional acoustic mode which naturally exists in the plasma with an additional ion population with a finite temperature, and, second, it yields an inaccurate Landau damping of the bulk ion acoustic mode. The reasons for these two failures of the model are described. In addition to this, a fluid model is presented that is capable of capturing both of these features that are missing in the approximative kinetic description, i.e., two (fast and slow) ion acoustic modes, and the corresponding Landau damping of both modes. © 2008 American Institute of Physics. [DOI: 10.1063/1.3036933]

I. INTRODUCTION

In the recent past there has been a lot of activity in the domain of theory and experiments dealing with the sheath in multi-ion plasmas.1–5 An earlier work on this issue6 provides us with a general expression for the Bohm criterion, while more recent theoretical7 and experimental8 works specify that, in a two-ion electron plasma, the sheath velocity of the two ion species have the same value that equals the hybrid “sound speed” in the literature.3

The behavior of the ion acoustic mode in the presence of an additional ion species may be drastically different compared to its behavior in a single ion plasma.7–10 In particular, the Landau damping is increased with the addition of a lighter ion species, which implies the possibility for controlling the mode behavior. Physically, this is equivalent to lowering the electron temperature and reducing the phase velocity, so that the damping is increased.7 Experimental observations of those phenomena are available in the classic Refs.8 and 9, and more recently in Ref. 10. We observe that these effects of the additional ions are just the opposite of the effects in the two (hot and cold) electron population case.11,12

We stress that each additional ion species introduces a new branch of acoustic oscillations in the system, complemented by various novel physical effects. However, in the context of the sheath theory, this domain of behavior of thermal ions has not yet been properly explored. This is the subject of the present work. In the case of a plasma with two ion species, one obtains two acoustic modes: one fast and one slow mode. As a result, the commonly used terms like the “system sound velocity” appear to be redundant and may become inapplicable in most plasmas.

II. COMPARISON BETWEEN THE COLD AND HOT IONS CASES

A. Cold ions

In order to see how thermal (kinetic) effects influence the oscillating plasma sheath, we first solve the dispersion equation for the ion acoustic (IA) mode propagating in a three component plasma consisting of electrons, and two cold ion species denoted by a and b,

\[ \frac{1}{k^2} = \frac{1}{[\omega - kv_{\text{IA}}(n_a/n_0)^{1/2}]^2} + \frac{(n_{b0}/n_0)(m_a/m_b)}{[\omega - kv_{\text{IA}}(n_a/n_0)^{1/2}]^2}. \]

(1)

Here, similar to Ref. 1, we have \( k = k \lambda_{\text{IA}} \), \( \omega \) is normalized to \( \omega_{\text{pe}} \), \( v_{\text{IA}} = v_{b0}/c_{\text{IA}} \) denotes the directed sheath currents/velocities of the two ion species, \( j = a, b \), and \( c_{\text{IA}}^2 = \kappa T_e/m_a \).

In the limit of large wavelengths and for negligible sheath currents, with the given normalization, the dispersion properties of the hybrid bulk IA mode are described by

\[ \omega^2 = k^2 [1 + m_a n_{b0}/(m_b n_{a0})]. \]

(2)

In physical units it reads \( \omega^2/k^2 = v_s^2 = (n_{b0}/n_{a0})\kappa T_e/m_a + (n_{a0}/n_{b0})\kappa T_e/m_b \), where \( v_s \) denotes the hybrid IA sound speed in plasmas with two ion species, called “the system sound speed” in the literature.3

The equation (1) for cold ions is solved numerically. First, similar to Ref. 1, we set \( v_{b0} = 0.009 \), \( v_{a0} = 0.011 \), which should correspond to the situation deep inside the plasma, far from the sheath, and we also choose \( m_a = 10 m_b \), and solve for \( n_{b0} = 0.2 n_{a0} \) in terms of \( k \). The result (for the ion acoustic mode) is presented in Fig. 1 with a dashed line, showing the tendency for the usual frequency saturation for large wave numbers. Going to larger values of \( k \) has no sense because of
physical reasons. Two additional complex-conjugate solutions for the sheath current driven modes, that also follow from Eq. (1), are about two orders of magnitude lower and are not of interest here (see Sec. III).

**B. Hot ions**

Now, for comparison, keeping the ion thermal effects, we rederive the dispersion equation using the kinetic theory. It reads

\[ \Delta(\omega, k) = 1 + \sum_{\alpha} \left( \frac{\omega_{p\alpha}^2}{k^2 v_{T\alpha}^2} \right) \left[ 1 - Z(\omega_{\alpha}, \nu_{\alpha}) \right] = 0. \] (3)

Here, \( \alpha = e, a, b, \ \omega_{\alpha} = \omega - kv_{\alpha} \), and \( Z(x) = \frac{x}{(2\pi)^{1/2}} \int dy e^{-y^2/2}/(x-y) \) is the plasma dispersion function, where \( x = \omega_{\alpha}/(kv_{\alpha}) \), \( y = \nu/
u_{T\alpha} \). For nonstreaming electrons and in the limit

\[ \nu_{T\alpha, b} \ll \omega/k \ll \nu_{T\alpha}, \] (4)

the standard expansions for \( Z \) are used, and the general dispersion equation (3) in that case becomes

\[
1 + \frac{\omega_{pe}^2}{k^2 v_{Te}^2} \left[ 1 + i(\pi/2)^{1/2} \frac{\omega}{k v_{Te}} \right] - \frac{\omega_{pa}^2}{k^2 v_{Ta}^2} - \frac{\omega_{pb}^2}{k^2 v_{Tb}^2} + 3 \frac{k^4 \omega_{pa}^4}{\omega_{pa}^4} + 4 \frac{k^4 \omega_{pa}^4}{\omega_{pa}^4} - i(\pi/2)^{1/2} \frac{\omega_{pa}^2}{k v_{Ta}} \exp\left[-\frac{\omega_{pa}^2}{2k^2 v_{Ta}^2}\right] - \frac{\omega_{pb}^2}{k^2 v_{Tb}^2} + 3 \frac{k^4 \omega_{pb}^4}{\omega_{pb}^4} - 4 \frac{k^4 \omega_{pb}^4}{\omega_{pb}^4} = 0. \] (5)

In the appropriate limits its real part yields Eq. (1), while the wave damping is given approximately by \( \gamma = -\Im \Delta(\omega, k)/[\partial \Re \Delta(\omega, k)/\partial \omega]_{\nu_{\alpha}=0} \). Here, \( \Im \Delta(\omega, k) \) and \( \Re \Delta(\omega, k) \) denote the imaginary and real parts of Eq. (5).

We have solved Eq. (5) numerically by setting \( T_e = T_b = T_e' = T_b' = T_e'' \), and for the same \( \nu_{\alpha}, n_{\alpha} \) as above. This result is also given in Fig. 1 (by full and dotted lines). The Landau damping has the maximum at \( k = 1 \), where \( |\gamma|/\omega = 0.25 \). Note that here and further in the text we present the absolute value of the Landau damping.

Observe a considerable change in the mode frequency. This may be of even greater importance than the damping because, compared to the cold ion limit, the phase speed is increased, which should be taken into account in modeling.

The presence of the second ion species affects both the mode frequency and the Landau damping, as shown in Fig. 2, where the hybrid IA mode frequency (normalized to \( \omega_{pe} \)) is presented in terms of the number density of the species \( b \), and for \( T_e = T_b = 30 \), and \( k = 0.1 \) (cf. Fig. 1). Here, in the beginning the Landau damping is increased by increasing the amount of the ion species \( b \), but this goes only up to \( n_b/n_a \) of about 6% when \( |\gamma|/\omega = 0.11 \). Note that for \( n_{b0} = 0 \) we have \( |\gamma| = 2 \times 10^{-4} \) (in units of \( \omega_{pe} \)), i.e., an 80 times lower damping compared with the case when \( n_{b0}/n_{a0} = 0.06 \). Clearly, the presence of the additional ion species significantly affects the damping of the mode.

For hotter ions (i.e., \( T_e = T_b = T_e'/10 \)), the ratio \( |\gamma|/\omega \) is increased, with the maximum at 0.22 at \( n_{b0}/n_{a0} = 0.2 \), as seen from Fig. 3. Here, the Landau damping grows with the addition of more \( b \)-ions, yet this increase in the damping saturates for the number density \( n_b \) reaching about 20%. Note that for \( n_{b0} = 0 \) it is \( |\gamma| = 4 \times 10^{-3} \), which is 12.5 lower compared with the case at \( n_{b0}/n_{a0} = 0.2 \).

Hence, the second ion species may drastically increase the Landau damping of the hybrid IA mode, and we see also that its relative effect is more pronounced for lower ion temperatures.

**III. SHEATH-CURRENT DRIVEN MODES**

As is well known, the presence of directed fluxes of plasma components implies the presence of extra collective modes. In our case these are the ion sheath-current (SC) driven, complex-conjugate solutions with frequencies of the order of \( \approx kv_{in} \), and with a growth rate that is approximately...
one order of magnitude lower. Their presence is obvious from the form of Eq. (1) and they are discussed below.

Increasing the directed ion velocities, the IA mode behavior is changed. This is visible from Fig. 4, where the velocities $v_{j0}$ are multiplied by a factor 100, becoming close to the values expected for the Bohm criterion, and for $n_b=0.2n_a$, $T_a=T_b=T_e/10$. Here, the normally negative solution for the hybrid IA mode (that is not discussed above in Fig. 1 as being of no interest) is strongly Doppler shifted that it becomes positive (line b). The absolute value of the Landau damping (the full line in Fig. 5) reaches its maximum at $k=0.7$ when $|\gamma|/\omega_a=0.16$.

On the other hand, the growing part of the two complex-conjugate SC solutions, in the $k$ domain presented in Fig. 5, has a growth rate that is about one order of magnitude below the mode frequency. It can easily be shown that for even larger values of $k \gg 15$ the instability vanishes and the real part of the SC mode frequency splits into two modes. However, this implies wavelengths below the electron Debye length and the model has no sense in that limit.

IV. ADDITIONAL ACUSTIC MODE

Apart from the demonstrated Landau damping and the modification of the frequency, the additional hot ion species further imply additional branches of acoustic oscillations. This can only be seen by solving the general dispersion equation (3) numerically, like in Refs. 7 and 9. Formally, the additional acoustic branches appear in the limit when the terms $k^2 v_{Tj}^2$ are kept finite in comparison with $\omega^2$. Therefore, in view of the condition (4), these additional acoustic branches are absent in the previous analysis and cannot be deduced from Eq. (5). Hence, the approximative kinetic approach describes the Landau damping but it is unable to describe the additional acoustic branch. However, a fluid model easily captures both of these features within the fluid theory. For Boltzmannian electrons and with the help of the Poisson equation, the derivation of the dispersion equation is straightforward yielding

$$1 + \frac{1}{k^2 \lambda_{de}^2} = \frac{\omega_{pa}^2}{\omega_{de} \omega_a - k^2 v_{Ta}^2} + \frac{\omega_{pb}^2}{\omega_{de} \omega_b - k^2 v_{Tb}^2}. \tag{6}$$

Here, $\omega_{j0} = \omega_{j0} + i \mu_{j0} k^2$. In the cold ions limit and for the given normalization, Eq. (6) becomes identical to Eq. (1). The term $\mu_{j0} = \lambda_{de} / (2\pi d_j)$, where $v_{Tj}^2 = v_{j0}^2 + v_{Tj}^2$, follows from the ion momentum equation of the form

$$\left( \frac{\partial}{\partial t} + v_{j0} \nabla \right) v_{j1} = - (q_j / m_j) \partial \phi_1 / \partial x - (\kappa T_j / n_{j0}) \partial n_{j1} / \partial x + \mu_{j0} \hat{p}^2 v_{j1} / \partial x^2,$$

where it has been introduced to describe the Landau damping. Such a fluid model has first been used in Ref. 15 and, more recently, in Refs. 14 and 16. It is convenient because it allows the use of the fluid theory, where the expressions can be analyzed more easily, as compared to the general kinetic expression (3) containing the plasma dispersion function expressed through the integral $Z(x) = [x/(2\pi)^{1/2}] \exp(-y^2/2)/(x-y)$. The term $d_j = \delta_j / \lambda$ in $\mu_{j0}$ gives the ra-
tio of the Landau attenuation length $\delta_j$ and wavelength $\lambda$. It is chosen in such a way that it is independent of the wave-length and the plasma density, and it depends on the ion temperature in a prescribed way. As is shown in Refs. 14 and 16, such a fluid model for the intrinsically kinetic Landau damping is, first, much more accurate than the approximative kinetic Landau damping obtained after the expansion of $Z(x)$ over the parameter $x$. Second, it gives the same damping as the exact kinetic Landau term. Third, it yields the analytical expression for the acoustic modes in two-ion electron plasmas with different temperatures of the two ion species [the alternative is numerically solving Eq. (3)].

All these features follow after adopting the following expression:

$$d_j = \delta_j/\lambda$$

$$= 0.275 000 700 8 + 0.042 007 307 780 9 \tau_j$$

$$+ 0.089 003 206 \tau_j^2 - 0.011 078 5 \tau_j^3 + 0.001 218 6 \tau_j^4. \quad (7)$$

Simple derivations$^{14}$ for an electron ion plasma yield the modeled fluid Landau damping as

$$|\gamma|/\omega = 1/(2\pi d). \quad (8)$$

The approximative$^{14}$ kinetic Landau damping is

$$|\gamma_{ap}|/\omega = (\pi/8)^{1/2}[(m_v/m_i)^{1/2} + (3 + \tau)^{1/2} \exp[-(3 + \tau)/2]]. \quad (9)$$

Finally, the exact kinetic Landau damping can be obtained only numerically. However, using the graph of the exact kinetic Landau damping from Ref. 11 one finds$^{14}$ that it may be expressed by the following polynomial:

$$|\gamma_{ex}|/\omega = 0.681 874 545 - 0.369 763 643 \tau$$

$$+ 0.090 345 088 805 5 \tau^2 - 0.012 034 27 \tau^3 + 0.000 752 396 767 7 \tau^4 - 0.000 018 \tau^5. \quad (10)$$

The above given statements about the accuracy of the expressions (8) and (9) can be directly checked by plotting the expressions (8)–(10) in terms of $\tau$, see Fig. 2 from Ref. 14.

It appears that, in the case of a two-ion electron plasma, the behavior of two acoustic modes is determined by the relation between $c_{sa}$ and $v_{Te}$. More details are available in Refs. 9 and 14. Equation (6) gives two (fast and slow) ion acoustic modes (that appear due to the fact that ions have some temperature), together with the corresponding Landau damping. As will be demonstrated below, this raises the question about the appropriateness of the “system sound velocity,” that is commonly used in the literature.

Using the same normalization as earlier, Eq. (6) becomes

$$1 + \frac{1}{k^2} = \left\{ (\omega - kv_{a0}/n_{a0}n_{e0})[\omega - kv_{a0}/n_{a0}] + i[kv_{a0}/(\pi d_a)][n_{e0}n_{a0}] - k^2 T_a n_{a0}/(T_n e_{a0}) \right\}^{-1}$$

$$+ \frac{n_{b0} m_a}{n_{a0} m_b} \left\{ (\omega - kv_{b0}/n_{b0}n_{a0})[\omega - kv_{b0}/n_{b0}] - k^2 m_p n_{b0}/(T_p n_{a0}) \right\}^{-1}. \quad (11)$$

Here, $v_{sa}^2 = (c_{sa}^2 + v_{Te}^2)/c_{sa}^2$. For comparison with previous cases, we solve Eq. (11) by taking the same parameters as in Fig. 1, i.e., $n_{a0} = 0.2n_{b0}$, $v_{a0} = 0.009$, $v_{b0} = 0.011$, $v_{sa} = 1.03$, $v_{sh} = 3.26$, $m_a = 10 m_b$. We have also chosen $T_e = 11 600$ K, $T_a = T_b = T_e/15$, and $m_p = 4 m_e$, so that $c_{sa} = 1547$ m/s. The two acoustic modes are presented in Fig. 6. The given sheath currents make negligible frequency shifts in the two acoustic modes. The corresponding Landau damping of the two modes, presented in Fig. 7, shows an obvious difference as
compared to Fig. 1. It is, first, lower by magnitude and, second, it shows no decrease for larger values of $k$. Hence, the decrease and saturation of the Landau damping, seen in previous figures, is clearly only due to the expansion of the plasma dispersion function, i.e., an artefact of the approximation used, and may not have any relevance to the real physical systems.

Similarly, when $n_{b0}$ is increased the frequency and the Landau damping of the upper (fast) IA mode is increased. However, for the slow IA mode these both parameters are reduced when $n_{b0}$ is increased. This is checked by setting $k = 0.3$ and for other parameters as above. The results are presented in Fig. 8 (for the two acoustic frequencies), and Fig. 9 for the absolute value of the Landau damping (multiplied by $10^5$). This behavior is in agreement with results existing in the literature.9,14

Setting the ion temperature to a larger value, i.e., $T_a = T_b = T_e/2$, the frequencies are increased by about a factor 1.5. However, the Landau damping is increased by a factor 100. This is presented in Figs. 10 and 11. It is seen also that the damping of the slow mode is now increased for larger values of $n_{b0}$.

V. SUMMARY

To summarize, the ion thermal effects are investigated in the sheath theory in multicomponent plasmas containing two ion species and electrons. Typically, this implies using the kinetic theory and we have demonstrated the differences between some results existing in the literature (obtained without ion thermal effects), and those obtained in the present analysis when the ion temperature effects are taken into account. The most important additional effect which follows from the finite ion temperature is the Landau damping, that is discussed in detail. However, the standard analysis implies the expansion of the plasma dispersion function, and as a result (a) some phenomena related to ion thermal effects are missed in the procedure, and (b) the mode behavior in some limits may show features that are not physical and that are
only due to the mentioned expansion. The most obvious example of (a) is the presence of an additional acoustic mode that can be seen only by solving the general kinetic dispersion Eq. (3) numerically.\textsuperscript{7,9} As for (b), such an artefact of the expansion is the saturation and decrease of the Landau damping, as demonstrated in Secs. II and III. In the case of hot ions, the expansion of the plasma dispersion function is not justified and an alternative approach is needed. We have demonstrated in Sec. IV that such a method exists within the framework of fluid theory.\textsuperscript{14–16} It is simple and suitable for analytical work, and at the same time is capable of capturing both effects, the additional ion acoustic mode and the Landau damping. In our earlier works\textsuperscript{14,16} we have shown that, in fact, quantitatively it describes the Landau damping much more accurately than the analysis which follows after the expansion of the plasma dispersion function.

Although suggested long ago,\textsuperscript{15} we observe that this fluid modeling of the Landau damping has remained unnoticed by researchers. In the present work, and in our recent publications\textsuperscript{14,16} we have shown that it can be successfully used in studies dealing with linear acoustic-type perturbations. In fact, this holds also for some low frequency modes in magnetized plasmas, like the drift wave, etc. This is seen from the general resonance condition in such plasmas $\omega - k_d v_{ij} = n \Omega_i$ ($n=0, \pm 1, \pm 2, \ldots$), which for a low frequency domain $\omega \ll \Omega_i$ implies just the resonance with the acoustic part of the mode. Hence, the model presented in this work can again be used. We stress that there already exists a number of different models and studies in the literature\textsuperscript{17–19} where fluid modeling of the Landau damping in magnetized plasmas has been performed.

We conclude that more care is needed whenever the standard expansion of the plasma dispersion function is used. This may directly be seen by comparing figures from Secs. II and III on one side, and those from Sec. IV on the other side. We stress also that even relatively small ion thermal effects imply the presence of an extra (slow) acoustic branch of ion oscillations, so that in fact two acoustic modes propagate in the plasma. The examples given in Sec. IV show that the frequencies of the two modes are not very distant even for relatively cool ions (in the given case the ion temperature of only about 1/15 of the electron temperature), yet the two modes have rather different behavior. Therefore, terms like “system sound velocity” may become inapplicable for most plasmas.

A possible extension of the results presented here could be the case of a two-ion electron plasma with one negative ion specie, and in particular the pair-ion electron plasma produced recently in the laboratory\textsuperscript{20} and discussed in the literature very extensively.\textsuperscript{21–25}

**ACKNOWLEDGMENTS**

This work has been completed during the visit of J.V. to the Department of High Energy Engineering Science, Kyushu University. The hospitality and excellent working conditions are gratefully acknowledged.

The results presented here are obtained in the framework of Project Nos. G.0304.07 (FWO-Vlaanderen), C 90205 (Prodex), GOA/2004/01 (K. U. Leuven), and the Interuniversity Attraction Poles Programme—Belgian State—Belgian Science Policy.

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