

DIVERGENT CESÀRO AND RIESZ MEANS OF JACOBI AND LAGUERRE EXPANSIONS

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ABSTRACT. We show that for δ below certain critical indices there are functions whose Jacobi or Laguerre expansions have almost everywhere divergent Cesàro and Riesz means of order δ .

1. INTRODUCTION

1.1. Orthogonal expansions. Suppose that (X, μ) is a positive measure space, $(\varphi_n)_{n=0}^\infty$ is an orthogonal subset of $L^2(X, \mu)$, and $h_n = \|\varphi_n\|_2^2$ for all $n \geq 0$. If f is a function on X for which all the products $f \cdot \overline{\varphi_n}$ are μ -integrable, then f has an orthogonal expansion

$$(1) \quad \sum_{n=0}^{\infty} c_n(f) h_n^{-1} \varphi_n(x)$$

with coefficients

$$(2) \quad c_n(f) = \int_X f(x) \overline{\varphi_n(x)} d\mu(x), \quad \forall n \geq 0.$$

1.2. Cesàro means. As described in Zygmund's book [16, pp. 76–77], the Cesàro means of order δ of the expansion (1) are defined by

$$(3) \quad \sigma_N^\delta f(x) = \sum_{n=0}^N \frac{A_{N-n}^\delta}{A_N^\delta} c_n(f) h_n^{-1} \varphi_n(x),$$

where $A_n^\delta = \binom{n+\delta}{n}$. Theorem 3.1.22 in [16] says that if the Cesàro means converge, then the terms of the series have controlled growth.

Lemma 1.1. *Suppose that $\lim_{N \rightarrow \infty} \sigma_N^\delta f(x)$ exists for some $x \in X$ and $\delta > -1$. Then*

$$|c_N(f) h_N^{-1} \varphi_N(x)| \leq C_\delta N^\delta \max_{0 \leq n \leq N} |\sigma_n^\delta f(x)|, \quad \forall N \geq 0.$$

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1.3. Riesz means. Hardy and Riesz [6] had proved a similar result for Riesz means. Recall that the Riesz means of order $\delta \geq 0$ are defined for each $r > 0$ by

$$(4) \quad S_r^\delta f(x) = \sum_{0 \leq n < r} \left(1 - \frac{n}{r}\right)^\delta c_n(f) h_n^{-1} \varphi_n(x).$$

Theorem 21 of [6] tells us how the convergence of $S_r^\delta f(x)$ controls the size of the partial sums $S_r^0 f(x)$.

Lemma 1.2. *Suppose that f has an orthogonal expansion and for some $\delta > 0$ and $x \in X$ its Riesz means $S_r^\delta f(x)$ converges to c as $r \rightarrow \infty$. Then*

$$|S_r^0 f(x) - c| \leq A_\delta r^\delta \sup_{0 < t \leq r+1} |S_t^\delta f(x)|.$$

Note that we can write

$$c_n(f) h_n^{-1} \varphi_n(x) = (S_n^0 f(x) - c) - (S_{n-1}^0 f(x) - c) = \mathbf{O}(n^\delta)$$

and obtain the same growth estimates as in Lemma 1.1.

Gergen [5] wrote formulae relating the Riesz and Cesàro means of order $\delta \geq 0$, from which the equivalence of the two methods of summation follows.

1.4. Uniform boundedness. Suppose there is a $1 < q \leq \infty$ for which $\varphi_n \in L^q(X, \mu)$ for all n . In addition, suppose that there is some positive number λ with

$$\|\varphi_n\|_q \geq cn^\lambda, \quad \forall n \geq 1.$$

The formation of the coefficient $f \mapsto c_n(f)$ is then a bounded linear functional on the dual of $L^q(X, \mu)$ with norm bounded below by a constant multiple of n^λ . The uniform boundedness principle implies that for p conjugate to q and each $0 \leq \varepsilon < \lambda$ there is an $f \in L^p(X, \mu)$ so that

$$(5) \quad c_n(f)/n^\varepsilon \rightarrow \infty \text{ as } n \rightarrow \infty.$$

1.5. Cantor-Lebesgue Theorem. The following argument is based on [16, Section IX.1]. Suppose we have a sequence of functions F_n on an interval in the real line with the asymptotic property

$$F_n(\theta) = c_n (\cos(M_n \theta + \gamma_n) + \mathbf{o}(1)), \quad \forall n \geq 0,$$

uniformly on a set E of finite positive measure, and with $M_n \rightarrow \infty$ as $n \rightarrow \infty$. Integrating $|F_n|^2$ over E gives

$$\begin{aligned} \int_E |F_n(\theta)|^2 d\theta &= |c_n|^2 \left(\int_E \cos^2(M_n \theta + \gamma_n) d\theta + \mathbf{o}(1) \right) \\ &= |c_n|^2 \left(\frac{|E|}{2} + \frac{e^{2i\gamma_n}}{4} \widehat{\chi}_E(2M_n) + \frac{e^{-2i\gamma_n}}{4} \widehat{\chi}_E(-2M_n) + \mathbf{o}(1) \right). \end{aligned}$$

The Riemann-Lebesgue Theorem [16, Thm. II.4.4] says that the Fourier transforms $\widehat{\chi}_E(\pm 2M_n) \rightarrow 0$ as $M_n \rightarrow \infty$. If we know that there is some function G for which $|F_n(\theta)| \leq G(n)$ uniformly on E for all n , then there is an $n_0 > 0$ for which

$$\frac{|E|}{4} |c_n|^2 \leq \int_E |F_n(\theta)|^2 d\theta \leq G(n)^2 |E|, \quad \forall n \geq n_0.$$

This shows that $|c_n| \leq 2G(n)$ for all $n \geq n_0$.

2. JACOBI POLYNOMIALS

2.1. **Notation.** Fix real numbers $\alpha \geq \beta \geq -1/2$, with $\alpha > -1/2$, and let μ denote the measure on $[-1, 1]$ defined by

$$d\mu(x) = (1 - x)^\alpha(1 + x)^\beta dx.$$

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n associated to the pair (α, β) as in Szegő's book [15]. Then $\left(P_n^{(\alpha,\beta)}\right)_{n=0}^\infty$ is an orthogonal subset of $L^2([-1, 1], \mu)$.

Equation (4.3.3) in [15] shows that the normalization terms $h_n^{(\alpha,\beta)} = \left\|P_n^{(\alpha,\beta)}\right\|_2^2$ satisfy

$$(6) \quad h_n^{(\alpha,\beta)} \sim c_{\alpha,\beta} n^{-1} \text{ as } n \rightarrow \infty.$$

The Jacobi polynomial expansion of $f \in L^1(\mu)$ is

$$\sum_{n=0}^\infty c_n(f) \left(h_n^{(\alpha,\beta)}\right)^{-1} P_n^{(\alpha,\beta)}(x),$$

with coefficients $c_n(f) = \int_{-1}^1 f(x) P_n^{(\alpha,\beta)}(x) d\mu(x)$. We take α and β as fixed and use $\sigma_N^\delta f(x)$ and $S_r^\delta f(x)$ to denote the Cesàro and Riesz means of this expansion, respectively.

2.2. **Asymptotics.** Theorem 8.21.8 in Szegő's book [15] gives the following asymptotic behaviour for the Jacobi polynomials.

Lemma 2.1. For $\alpha \geq \beta \geq -1/2$ and $\varepsilon > 0$ the following estimate holds uniformly for all $\varepsilon \leq \theta \leq \pi - \varepsilon$ and $n \geq 1$:

$$(7) \quad P_n^{(\alpha,\beta)}(\cos \theta) = n^{-1/2} k(\theta) \cos(M_n \theta + \gamma) + \mathbf{O}\left(n^{-3/2}\right).$$

Here $k(\theta) = \pi^{-1/2} (\sin(\theta/2))^{-\alpha-1/2} (\cos(\theta/2))^{-\beta-1/2}$, $M_n = n + (\alpha + \beta + 1)/2$, and $\gamma = -(\alpha + 1/2)\pi/2$.

From Egoroff's theorem and Lemma 1.1 we can say that if $\sigma_N^\delta f(x)$ converges on a set of positive measure in $[-1, 1]$, then there is a set of positive measure E on which

$$(8) \quad \left|c_n n^{(1/2)-\delta} (\cos(M_n \theta + \gamma) + \mathbf{O}(n^{-1}))\right| \leq A$$

uniformly for $\cos \theta \in E$. The argument of subsection 1.5 shows that

$$(9) \quad \left|c_n n^{(1/2)-\delta}\right| \leq A, \quad \forall n \geq 1.$$

2.3. **Norm estimates.** Next we recall the calculation of Lebesgue norms of Jacobi polynomials, according to Markett [10] and Dreseler and Soardi [4]. Equation (2.2) in [10] gives the following lower bounds on these norms.

Lemma 2.2. For real numbers $\alpha \geq \beta \geq -1/2$, with $\alpha > -1/2$, $1 \leq q < \infty$, and $r > -1/q$,

$$\left(\int_0^1 \left|P_n^{(\alpha,\beta)}(x) (1-x)^r\right|^q dx\right)^{1/q} \sim \begin{cases} n^{-1/2} & \text{if } r > \alpha/2 + 1/4 - 1/q, \\ n^{-1/2} (\log n)^{1/q} & \text{if } r = \alpha/2 + 1/4 - 1/q, \\ n^{\alpha-2r-2/q} & \text{if } r < \alpha/2 + 1/4 - 1/q. \end{cases}$$

2.4. Main result. There are critical indices, as used in [11],

$$p_c = \frac{4(\alpha + 1)}{(2\alpha + 3)} \quad \text{and its conjugate } p'_c = \frac{4(\alpha + 1)}{(2\alpha + 1)}.$$

Taking $r = \alpha/q$ in Lemma 2.2 we have that

$$(10) \quad \left\| P_n^{(\alpha, \beta)} \right\|_{L^q(\mu)} > \left(\int_0^1 \left| P_n^{(\alpha, \beta)}(x) \right|^q (1-x)^\alpha dx \right)^{1/q} \sim n^{\alpha - 2\alpha/q - 2/q}$$

for $\alpha/q < \alpha/2 + 1/4 - 1/q$. This last inequality can be rewritten as

$$(11) \quad q > \frac{4(\alpha + 1)}{2\alpha + 1} = p'_c.$$

We can now prove that below the critical index there are functions with almost everywhere divergent Cesàro and Riesz means.

Theorem 2.3. For real numbers $\alpha \geq \beta \geq -1/2$, with $\alpha > -1/2$,

$$1 \leq p < p_c = \frac{4(\alpha + 1)}{(2\alpha + 3)}, \quad \text{and } 0 \leq \delta < \frac{(2\alpha + 2)}{p} - \frac{(2\alpha + 3)}{2},$$

there is an $f \in L^p(\mu)$, supported in $[0, 1]$, whose Cesàro means $\sigma_N^\delta f(x)$ and Riesz means $S_r^\delta f(x)$ are divergent almost everywhere on $[-1, 1]$.

Proof. Let q be conjugate to p , so that $1/p = (q - 1)/q$. Suppose that

$$\delta < \frac{(2\alpha + 2)}{p} - \frac{(2\alpha + 3)}{2} = \frac{(2\alpha + 2)(q - 1)}{q} - \frac{(2\alpha + 3)}{2} = \alpha + \frac{1}{2} - \frac{(2\alpha + 2)}{q}.$$

Then

$$\delta - \frac{1}{2} < \alpha - \frac{(2\alpha + 2)}{q} = \alpha - \frac{2\alpha}{q} - \frac{2}{q},$$

which is the exponent of n in the inequality (10). Now apply the argument given in subsection 1.4. The norms of the Jacobi polynomials in Lemma 2.2 are calculated over $[0, 1]$ and so we can find f in $L^p([-1, 1], \mu)$, supported on $[0, 1]$, for which the coefficients satisfy

$$c_n(f)/n^{\delta - 1/2} \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

Combine this with Lemmas 1.1 and 1.2 and the argument around inequality (9) to see that for this f both $\sigma_N^\delta f(x)$ and $S_r^\delta f(x)$ are divergent almost everywhere. This argument follows the methods used in [11, 8, 7, 12]. \square

2.5. Remarks. Convergence results above the critical index are contained in the work of Bonami and Clerc [1], Colzani, Taibleson and Weiss [3], and Chanillo and Muckenhoupt [2]. In particular, in [2, Thm. 1.4] it is shown that for

$$1 \leq p < p_c = \frac{4(\alpha + 1)}{(2\alpha + 3)} \quad \text{and } \delta = \frac{(2\alpha + 2)}{p} - \frac{(2\alpha + 3)}{2},$$

the maximal operator $f \mapsto \sup_{N \geq 0} |\sigma_N^\delta f(x)|$ is of weak type (p, p) .

For $\delta = 0$ and $p = p_c$, divergence was proved in [11].

3. LAGUERRE FUNCTIONS

3.1. **Notation.** For each $\alpha > -1$ let μ_α be the measure on $[0, \infty)$ defined by

$$d\mu_\alpha(x) = e^{-x}x^\alpha dx.$$

We denote by $L_n^{(\alpha)}(x)$ the Laguerre polynomial of degree n , as in [15, Chpt. 5]. The $L^2(\mu_\alpha)$ -norms of these satisfy the identity

$$h_n^{(\alpha)} = \|L_n^{(\alpha)}\|_{L^2(\mu_\alpha)}^2 = \Gamma(\alpha + 1) \binom{n + \alpha}{n} \sim n^\alpha.$$

Fejér’s formula [15, Thm. 8.22.1] gives the asymptotic properties of these polynomials. For each $\alpha > -1$ and $0 < \varepsilon < \omega$,

$$(12) \quad L_n^{(\alpha)}(x) = \frac{e^{x/2}}{\pi^{1/2}x^{\alpha/2}}n^{\alpha/2-1/4} \cos\left(2(nx)^{1/2} - \alpha\pi/2 - \pi/4\right) + \mathbf{O}\left(n^{\alpha/2-3/4}\right),$$

uniformly in $x \in [\varepsilon, \omega]$. The corresponding normalized functions are

$$(13) \quad \mathcal{L}_n^\alpha(x) = \sqrt{\frac{\Gamma(n + 1)}{\Gamma(n + \alpha + 1)}} e^{-x/2}x^{\alpha/2}L_n^{(\alpha)}(x), \quad \forall x \geq 0, n \geq 0.$$

These provide an orthonormal subset of $L^2([0, \infty))$, where the half line carries Lebesgue measure.

3.2. **Norm estimates.** Markett [9, Lemma 1] has calculated the Lebesgue norms of the Laguerre functions, for $\alpha > -1/2$,

$$(14) \quad \|\mathcal{L}_n^\alpha\|_q \sim \begin{cases} n^{1/q-1/2}, & \forall 1 \leq q < 4, \\ n^{-1/4}(\log n)^{1/4}, & \text{if } q = 4, \\ n^{-1/q}, & \forall 4 < q \leq \infty. \end{cases}$$

3.3. **Divergence result.**

Theorem 3.1. *If $\alpha > -1/2$, $p > 4$ and $0 < \delta < 1/4 - 1/p$, then there is a function $f \in L^p(0, \infty)$ whose Laguerre expansion*

$$\sum_{n=0}^\infty c_n(f)\mathcal{L}_n^\alpha(x)$$

has Cesàro and Riesz means of order δ which diverge almost everywhere.

Proof. Suppose that the expansion $\sum_{n=0}^\infty c_n(f)\mathcal{L}_n^\alpha(x)$ is either Cesàro or Riesz summable of order δ on a set of positive measure in $[0, \infty)$. Then Lemma 1.1 or Lemma 1.2 implies that

$$(15) \quad c_n(f)\mathcal{L}_n^\alpha(x) = \mathbf{O}(n^\delta)$$

on a set of positive measure. When equations (12) and (13) are combined with the argument of subsection 1.5 we find that

$$(16) \quad c_n(f) = \mathbf{O}(n^{\delta+1/4}).$$

The case when $\delta = 0$ is Lemma 2.3 in Stempak’s paper [14]. Suppose that

$$\frac{1}{q} - \frac{1}{2} > \delta + \frac{1}{4},$$

so that $\delta < \frac{1}{q} - \frac{3}{4} = \frac{4-3q}{q}$. If $\frac{1}{q} = 1 - \frac{1}{p}$, then this inequality is $\delta < \frac{1}{4} - \frac{1}{p}$. The argument of subsection 1.4 shows that if $p > 4$ and $\delta < 1/4 - 1/p$, then there is a function $f \in L^p(0, \infty)$ for which the inequality (16) fails,

$$c_n(f)/n^{\delta+1/4} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

The Laguerre expansion of this function has Cesàro and Riesz means of order δ which diverge almost everywhere. \square

3.4. Remarks. There is an extensive treatment of almost everywhere convergence results for Laguerre expansions in [13]. In particular, [13, Thm. 1.20] implies that if $p > 4$ and $\delta \geq 1/4 - 1/p$, then all $f \in L^p(0, \infty)$ have almost everywhere convergent Cesàro means of order δ .

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