CONGRUENT RISK MEASURES IN A BANKING FIRM

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ABSTRACT
This paper examines the use of risk measures within a banking firm for the purposes of capital allocation and risk-adjusted performance measurement. The paper explores a set of risk measures to determine their congruency with the risk preference function of the centre of the bank, which is assessed to be one characterised by non-satiety, risk-aversion and positive skewness in the distribution of returns – risk aversion on the part of bank stakeholders. The paper also examines the question of whether risk measures used internally in a bank for capital allocation and risk-adjusted performance measurement need to be coherent, in terms of the axioms of coherency developed by Artzner et al (1999).
1. **INTRODUCTION**

This paper examines the use of risk measures within a banking firm for the purposes of capital allocation and performance measurement. The paper explores a set of risk measures to determine their congruency with the risk preference function of the centre of the bank, which is assessed to be one characterised by non-satiety, risk-aversion and positive skewness in the distribution of returns – risk aversion on the part of bank stakeholders.

We determine that portfolios that are dominant by third-order stochastic dominance principles are compatible with the risk preference function of the centre. The paper provides a framework from which we assess which risk measures are compatible with the risk preferences of the centre, such that they would lead managers to select the portfolios that the centre would have them select if it was aware of the full opportunity set available to managers. We categorise risk measures in terms of their implicit risk attitudes, and apply the framework to five risk measurement candidates.

Artzner et al (1999) present and justify a set of four desirable structural properties for measures of risk, which they argue should hold for any risk measure which is to be used to effectively regulate or manage risks. They call measures that satisfy these properties ‘coherent’. The four axioms that characterise coherent risk measures are translation invariance, monotonicity, positive homogeneity and subadditivity. We find that risk measures that are incentive-compatible in terms of stochastic dominance principles are not necessarily coherent. This leads us to question if risk measures need to be coherent, in keeping with Artzner et al (1999), in order to be incentive-compatible. We examine this question and determine that risk measures that fail to meet at least two coherency axioms – positive homogeneity and subadditivity – may lead managers to make suboptimal investment decisions. This is despite the fact that these measures conform to risk-ordering according to stochastic dominance principles.
The paper is structured as follows. Section 2 examines the risk ranking of bank credit portfolios in terms of stochastic dominance principles. Section 3 examines a general class of risk measures in term of stochastic dominance criteria, while Section 4 considers the compatibility of five downside risk measures from a framework of stochastic dominance. Section 5 examines the coherency of the risk measures, in keeping with Artzner et al (1999). Section 6 concludes the paper.

2. RISK RANKING CRITERIA

In situations where there is complete information on preferences, a complete ordering of alternative investments can be undertaken based on the expected utility function of the investor. In this setting, those portfolios with the highest expected utility are the dominant portfolios. It seems, however, that few banking firms have the willingness or means by which to parameterise their own utility function - perhaps reflecting the dominance of regulatory constraints over the risk preferences of owners and managers. With incomplete information on the exact form of the utility function for the banking firm, we can only determine a partial ordering of the available investments.

Stochastic dominance is a generalisation of utility theory that eliminates the problem of having to explicitly specify the utility function of the investor.\(^1\) The central idea of stochastic dominance is that the decision problem can be simplified by sorting out and eliminating dominated alternatives. Stochastic dominance converts the probability distribution of an investment into a cumulative probability curve, which is used to determine the superiority of one investment over another. Stochastic dominance criteria provide a set of rules for making choices among risky assets consistent with the preferences of broad classes of utility functions, obviating the need to know the precise functional characterisation of the objective function. Different orders of stochastic dominance correspond to different classes of utility function. We outline the selection criteria that apply to each order of stochastic dominance below, and

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\(^1\) The contemporary notion of stochastic dominance has its roots in papers by Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970).
assess the applicability of the assumptions for each order for the risk preferences of a banking firm.

2.1. First-order Stochastic Dominance

First-order stochastic dominance (FSD) provides a rule for rank-ordering risky portfolios in a manner consistent with the preferences of investors who prefer more wealth to less. A portfolio stochastically dominates another portfolio by FSD if investors receive greater wealth from the portfolio in every ordered state of nature. This means the only requirement for FSD is that utility functions are increasing: FSD does not encompass the risk attitude of the investor.

Let F and G represent the cumulative probability distributions of the returns for portfolios X and Y, and let $U(w)$ refer to the utility of w dollars of wealth. Under the FSD selection rule, portfolio X will stochastically dominate portfolio Y if

$$F_X(w) \leq G_Y(w)$$

for all $w$ with at least one strict inequality.\(^2\) Alternatively,

$$[G_Y(w) - F_X(w)] \geq 0$$

for all $w$ with at least one strict inequality. This means the cumulative probability distribution for portfolio Y always lies to the left of the cumulative distribution for portfolio X. Further, for investors to prefer more wealth to less, the utility function must be increasing monotonically. This implies a positive first derivative for the utility function:

$$U'(w) > 0.$$ 

Increasing wealth preference can be considered universal for all utility functions and representative of the behaviour of the banking firm. Indeed, as mentioned, this includes investors who are risk-seekers, risk-aversers and those who are risk-neutral.

As such, a large proportion of the given set of investment alternatives will be members of the FSD admissible set, restricting the practical applicability of the FSD selection rule.

2.2. Second-order Stochastic Dominance

Second-order stochastic dominance (SSD) assumes that in addition to increasing wealth preference, investors are risk-averse. Risk aversion can be defined where the utility function of an investor is increasing and concave, implying a positive first derivative and a negative second derivative for the utility function:

\[ U'(w) > 0 \text{ and } U''(w) < 0 \]

Under the assumption of risk aversion, the expected utility of a risky investment portfolio is less than the utility of the expected outcome.

Under the SSD selection rule, portfolio X will dominate portfolio Y if

\[ \int_{-\infty}^{w} [G_y(w) - F_x(w)] \, dw \geq 0 \]

for all \( w \) with at least one inequality.\(^3\) This means that in order for portfolio X to dominate portfolio Y for all risk-averse investors, the accumulated area under the cumulative probability distribution of Y must be greater than the accumulated area for X, below any given level of wealth. Unlike FSD, this implies that the cumulative density functions can cross. Further, a necessary condition for SSD of portfolio X over Y is that the expected value of portfolio X is greater than or equal to the expected value of Y.

The assumption that investors are risk averse provides a stronger utility function constraint than under FSD, and as such, the SSD admissible set is smaller than that under the FSD criterion.  

2.3. Third-order Stochastic Dominance

Third-order stochastic dominance (TSD) corresponds to the set of utility functions where:

\[ U'(w) > 0, \quad U''(w) < 0 \quad \text{and} \quad U'''(w) > 0 \]

The addition of a negative third derivative for the utility function requires the investor to prefer positive skewness in the distribution of portfolio returns (upside returns will have a larger magnitude than downside returns, indicating greater probability in the right tail of the distribution). Using data on the rates of return of mutual funds, Levy (1998) provides empirical evidence that supports the hypothesis that most investors prefer positive skewness and dislike negative skewness. From the perspective of a banking firm, a preference for positive skewness can be interpreted as an unwillingness to accept a small and almost certain gain in exchange for a remote possibility of the bank defaulting on its debt obligations.

Under the TSD selection rule, portfolio X will dominate portfolio Y if and only if the following conditions hold:

\[
\int_{-\infty}^{w} \int_{-\infty}^{t} \left[ G_x(w) - F_x(w) \right] dw \, dt \geq 0, \quad \text{and} \quad E_F(x) \geq E_G(x)
\]

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4 This has been empirically verified by Levy and Sarnat (1970) and Levy and Hanoch (1970).
for all \( w \) with at least one inequality.\(^7\) This means a preference for one portfolio over another by TSD may be due to the preferred investment having a higher mean, a lower variance or a higher positive skewness.\(^8\)

In addition to positive skewness preference, a rationale for a positive third derivative for the investor’s utility function is decreasing absolute risk aversion, meaning the higher the wealth of the investor, the smaller the risk premium that the investor would be willing to pay to insure a given loss. While this may be the case for bank owners, this aspect of TSD is less relevant in the current context than the preference for positive skewness. The unwillingness for the investor to accept a small and almost certain gain in exchange for a remote possibility of ruin is a property of TSD that directly conforms to the risk preferences of bank regulators and bank creditors.

Bawa (1975) shows that for the entire class of distribution functions and for the class of decreasing absolute risk-averse utility functions, the TSD rule is the optimal selection rule when distributions have equal means. While in cases where distributions have unequal means there is no known selection rule that satisfies both necessary and sufficient conditions for dominance, Bawa shows that the TSD rule may be used as a reasonable approximation to the optimal selection rule for the entire class of distribution functions.

TSD represents the most applicable criteria for ranking alternative investment portfolios in the bank setting given the TSD dominant portfolio embodies risk aversion and positive skewness preference. If bank owners seek to preserve bank franchise value, their utility function will display risk aversion \((U''(w) < 0)\). If bank creditors and regulators are concerned with the size of losses in the event of the bank becoming insolvent, they will demonstrate a preference for positive skewness in bank returns \((U'''(w) > 0)\). TSD also applies to the entire shape of the distribution function of bank returns, and thus allows for non-normality in returns. This is important given

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\(^7\) For a proof, see Levy (1998) p.92-96. The symbol \( t \) arises from the thrice integration of the expression \( E_w[U(w_F)] - E_w[U(w_G)] \geq 0 \). It indicates that the cumulative of the cumulative of the cumulative distribution function of \( F \) lies above \( G \). See Heyer (2001).

the non-normal distribution of returns that characterise many bank portfolios, and in particular, loan portfolios. In contrast, the popular mean-variance criterion is only accurate for ranking portfolios that are normally distributed.9

3. COMPATIBILITY OF RISK MEASURES WITH STOCHASTIC DOMINANCE CRITERIA

In a recent survey of the risk measurement literature, Albrecht (2003) subsumes risk measures into two broad categories: (1) risk as the magnitude of deviations from a target, and (2) risk as a measure of the overall seriousness of possible losses. In the second category, risk is regarded as the capital that must be added to a position to make it riskless. The two categories are linked in the sense that the first can be used as a basis for determining the capital requirement in the second.

With respect to the first category, risk measures can be two-sided or one-sided. Since the work of Markowitz (1952), variance (standard deviation) has been the traditional two-sided measure of risk. The theoretical arguments against using the mean-variance approach for ranking investments centre on the properties of a quadratic utility function (which exhibits increasing and absolute risk aversion) and a normal distribution of returns. These two-sided risk measures assign the same weight to both positive and negative deviations from the expected value, which contradicts the notion that investors view risk as negative deviations from an expected value. Further, variance does not capture kurtosis in the underlying distribution of returns, which is needed if investors wish to incorporate the risk of low probability/high loss events in their assessment of investment alternatives.

*Lower partial moments* (LPM) are a general class of risk measures where risk is measured in terms of negative deviations from a predetermined loss threshold or target rate of return. For continuous distributions, LPMs are measured as follows:

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9 See Elton and Gruber (1995), p.244-5.
\[ \text{LPM} (n, t) = \int_{-\infty}^{t} (t-x)^n f(x) \, dx \quad n > 0, \]

where \( t \) is the target rate of return, \( x \) are the outcomes of the probability distribution and \( f(x) \) is its density function. The exponential variable \( n \) is the degree of the lower partial moment, and represents the weight that an investor places on negative deviations from the target. The exponential variable thus allows the LPM to describe below-target risk in terms of the risk tolerance of the investor.

Bawa (1975) provides a proof of the mathematical relationship between lower partial moments and stochastic dominance for risk tolerance levels of \( n = 0, 1 \) and 2, with higher orders of \( n \) corresponding to greater risk aversion on the part of investors. Fishburn (1977) also provides theoretical support for using lower partial moments to capture the utility functions of specific investors. Specifically, these authors show that LPM\(_0\) is applicable to all utility functions showing non-satiety \((u' > 0)\), and that this is analogous to first-order stochastic dominance rules. Further, they show that LPM\(_1\) is consistent with all risk-averse functions \((u' > 0, u'' < 0)\), and that this is analogous to second-order stochastic dominance rules, while LPM\(_2\) is consistent for all risk-averse functions displaying skewness preference \((u' > 0, u'' < 0, u''' > 0)\), and this corresponds to third-order stochastic dominance rules.

The findings of Bawa (1975) and Fishburn (1977) are significant in our search for risk measures that are compatible with the risk preference function of the centre of the bank because they indicate consistency between specific risk measures and \(n^{th}\) order stochastic dominance criteria. The strong relationship between risk measures based on lower partial moments and stochastic dominance concepts is significant given that stochastic dominance criteria apply to a general class of utility functions and make no assumptions regarding the distribution of portfolio returns. Our earlier conclusion that third-order stochastic dominance (TSD) represents the most applicable criteria for ranking alternative investment portfolios in the bank setting suggests that lower partial moments of order \( n > 1 \) may be a more relevant family of risk measures for the centre of the bank. The key research question is would risk measures based on higher order lower partial moments lead managers/agents in banks to select the portfolios that the
centre would have them select in the presence of perfect information? In order to address this question we will shortly examine a range of risk measures within the general class of lower partial moments.

Drawing on the findings of Bawa (1975) and Fishburn (1977), we begin with the following definition:

A risk measure $\rho(X)$ is consistent with $n^{th}$ order stochastic dominance, and consequently expected utility maximisation, when portfolios can be ranked by $n^{th}$ order stochastic dominance. Specifically, for portfolios $X_1$ and $X_2$:

$$X_1 \text{ SD}(n) X_2 \rightarrow \rho(X_1) \leq \rho(X_2)$$

This means that if one portfolio $X_1$ stochastically dominates another portfolio $X_2$, the risk measure for $X_1$ will be lower than for $X_2$ and the expected utility deriving from portfolio $X_1$ will exceed that from $X_2$. Further, a risk measure that is consistent with $(n+1)^{th}$ order stochastic dominance is also consistent with $n^{th}$ order stochastic dominance. This means a risk measure consistent with higher-order stochastic dominance will be more applicable than a risk measure consistent with lower-order stochastic dominance.

Next, we establish criteria under which a risk measure $\rho(X)$ incorporates losses in the tail of a distribution beyond a specified threshold. Using the definition of the Bank for International Settlements (2000), ‘tail risk’ arises when the probability of returns for one investment has a greater risk of larger losses than another, where both distributions have equal means. Tail risk is applicable whenever bank stakeholders are concerned with the size of losses in the event that losses exceeded a predetermined level. We establish that bank creditors, depositors and regulators are more likely to be concerned with the size of losses in the event of default (tail risk), whereas managers and owners are likely to be more concerned with the probability of losses given their personal losses are linked to default itself, and are invariant to the size of losses.

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10 See Levy (1998), theorem 3.2, for a proof.
With respect to a risk measure $\rho(X)$ and the degree of tail risk in the distribution of a portfolio, we put forward the following proposition:

*If beyond a specified loss threshold, a portfolio $X_1$ has lower expected losses than another portfolio $X_2$, then the risk measure $\rho(X_1)$ should be lower than the risk measure $\rho(X_2)$.*

This requires that our risk measure(s) should differentiate among portfolios in terms of their tail risk. It remains to establish how a risk measure can be categorised in terms of its ability to capture specific degrees of tail risk in a loss distribution.

First, let us consider the most restrictive case. If an investor is concerned only that the probability of loss for one portfolio $X_1$ is less than the probability of loss for another portfolio $X_2$, then the cumulative distribution function for portfolio $X_1$ will always lie to the right of the cumulative distribution function for portfolio $X_2$. In other words, the cumulative distribution functions for the portfolios under consideration should not cross. As discussed earlier, this implies a positive first derivative for the utility function of the investor.

Under these conditions we can say for a risk measure $\rho(X)$, at a given loss threshold $(t)$, that $\rho(X_1) < \rho(X_2)$ if:

\[ F_1(x) \leq F_2(x), \text{ for all } x, x \leq t \]

where $F_1(x)$ and $F_2(x)$ are the cumulative distribution functions for $X_1$ and $X_2$. Notably, this is consistent with portfolio $X_1$ dominating portfolio $X_2$ in terms of first-order stochastic dominance principles (FSD). Risk measures that fit into this category we denote ‘Type 1’ risk measures.

Next, consider the case of an investor who is concerned not only with the probability of losses, but also the size of losses should a certain loss threshold $(t)$ be exceeded. If an investor is concerned that the average losses for portfolio $X_1$ be less than the
average losses for portfolio $X_2$, for a given loss threshold ($t$), then the accumulated
area under the cumulative distribution function for portfolio $X_1$ must be less than the
accumulated area under the cumulative distribution function for portfolio $X_2$. Unlike
the first case above, this means that the cumulative density functions can cross, and
this in turn is consistent with portfolio $X_1$ dominating portfolio $X_2$ according to second
order stochastic dominance principles (SSD). Further, in addition to increasing wealth
preference, this implies the investor is risk-averse (negative second derivative for the
utility function).

Under these conditions we can say for a risk measure $\rho(X)$, at a given loss threshold
($t$), that $\rho(X_1) < \rho(X_2)$ if:

$$\int_{-\infty}^{t} (t - x) f_1(x) \, dx \leq \int_{-\infty}^{t} (t - x) f_2(x) \, dx \quad \text{for all } x, x \leq t$$

where $f_1(x)$ and $f_2(x)$ are the density functions of $X_1$ and $X_2$. Risk measures that fit into
this category we denote ‘Type 2’ risk measures.

Finally, consider an investor who perceives a low probability of a large loss to be
riskier than a high probability of a small loss, even when expected losses are the same.
In addition to increasing wealth preference and risk aversion on the part of the
investor, this condition also encapsulates a preference for positive skewness in the
portfolio distribution, and is consistent with a negative third derivative for the utility
function of the investor.

Drawing on this, let two portfolios, $X_1$ and $X_2$, have equal means and equal average
losses beyond a given loss threshold ($t$), but portfolio $X_2$ has a small probability of
large losses beyond the threshold while portfolio $X_1$ has a larger probability of small
losses beyond the threshold. If an investor displays a preference for positive skewness
in the distribution of portfolio returns, we can say for a risk measure $\rho(X)$, that $\rho(X_1) <
\rho(X_2)$ if the following holds:
\[
\int_{-\infty}^{t} (t-x)^{(n-1)} f_1(x) \, dx \leq \int_{-\infty}^{t} (t-x)^{(n-1)} f_2(x) \, dx \quad \text{for all } x, x \leq t
\]

where \( f_1(x) \) and \( f_2(x) \) are the density functions of \( X_1 \) and \( X_2 \) and \( n > 2 \).

This condition employs the lower partial moment of degree \((n-1)\) to penalise large deviations from the loss threshold more than smaller deviations from the loss threshold. This differs from the previous condition, which implicitly assumed that investors have a linear response to losses beyond the threshold. Where \( n = 2 \), this condition is equal to the previous condition. When \( n = 3 \), this condition places a quadratic penalty on deviations below the loss threshold\(^{12}\), and is consistent with portfolio \( X_1 \) dominating portfolio \( X_2 \) according to third order stochastic dominance principles (TSD). Risk measures that fit into this category, where \( n = 3 \), we denote ‘Type 3’ risk measures.

We summarise each risk measure category as follows:

<table>
<thead>
<tr>
<th>Risk Measure Category</th>
<th>Characteristics</th>
<th>Stochastic Dominance Compatibility</th>
<th>Implicit Risk Attitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>The investor is concerned with probability of loss beyond a given threshold</td>
<td>FSD</td>
<td>Non-satiety</td>
</tr>
<tr>
<td>Type 2</td>
<td>Investor is concerned with probability of losses and the average size of losses beyond a given threshold</td>
<td>SSD</td>
<td>Non-satiety and risk aversion</td>
</tr>
<tr>
<td>Type 3</td>
<td>Investor is concerned with probability of losses, the average size of losses and larger deviations more than smaller deviations from a given threshold</td>
<td>TSD</td>
<td>Non-satiety, risk aversion and positive skewness</td>
</tr>
</tbody>
</table>

\(^{12}\) Higher order powers \((n > 3)\) place even larger penalties on wider deviations from the loss threshold. For example, \( n = 4 \) places a cubic penalty on deviations below the loss threshold.
4. RISK MEASURES

In this section we examine a number of downside risk measures to assess their compatibility with the principles established in the preceding section. The downside risk measures under examination are based on statistics of the loss distribution of a portfolio over some predetermined time horizon. The selected measures, which are somewhat contemporary measures of downside risk, are the shortfall probability, value at risk (VaR), expected shortfall, first-order lower partial moment (LPM1) and the second-order lower partial moment (LPM2)\(^{13}\).

4.1. Shortfall probability

Shortfall probability is the probability that the return on a portfolio will fall below the prespecified target level. Shortfall probability is measured by the zero-order lower partial moment (LPM\(_0\)), which is the integral of the unweighted return distribution. It is defined as follows:

\[
\text{LPM} (0, t) = \int_{-\infty}^{t} (t-x)^0 f(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx = F(t)
\]

where \(F(t)\) is the cumulative density function for expected returns below the target \((t)\). Shortfall probability is the relevant risk measure for an investor who is only interested in the probability of falling short of the prespecified target return, ignoring the extent or severity of the shortfall should it eventuate. Fishburn (1977) demonstrates that this measure is consistent with a risk-seeking utility function; specifically the order \(n < 1\) characterises an investor who is willing to gamble at fair odds in an attempt to minimise the extent to which returns fall short of the target.

\(^{13}\) The second-order lower partial moment is also known as the shortfall variance.
4.2. Value-at-Risk

Value-at-risk (VaR) is defined as the loss that will not be exceeded over a certain time period with a specified confidence level ($\alpha$). Put differently, the VaR of an investment is the loss that will be exceeded only with a given probability ($1 - \alpha$) over a specified measurement period. VaR is closely related to shortfall probability through the cumulative distribution function. If the VaR is designated as the benchmark for measuring shortfall, the probability that losses exceed the VaR level corresponds to the shortfall risk measure.\textsuperscript{14} In terms of our earlier defined criteria, VaR and shortfall probability correspond to Type 1 risk measures.

If $V$ is the value of an investment at the end of the designated measurement period, we define $V_p$ such that

$$ Prob (V \leq V_p) = \int_{-\infty}^{V_p} dF(V) = 1 - \alpha $$

where $F(V)$ is the cumulative distribution function of $V$. Thus the value of the investment will be below $V_p$ with a probability of ($1 - \alpha$). In defining the VaR, it remains to determine the benchmark such that outcomes below this benchmark are regarded as losses. This benchmark, for example, may be the initial value of the investment or the expected value of the investment. If the expected value of the investment is determined to be the benchmark, the VaR is defined as follows:

$$ VaR_\alpha = E(V) - V_p $$

In the bank loan setting, the expected value of the loan can be considered the appropriate benchmark because this determines the cut-off point for expected losses. For example, if the maximum value of a loan is determined to be $100 and the expected value of the loan is $99, expected losses are equal to $1. The expected value of $99 then forms the benchmark for determining unexpected losses. To further the

\textsuperscript{14} See Schroder (1997) for a mathematical derivation.
example, if the distribution of loan values indicates there is a 1% probability that the value of the loan will be lower than $60 (ie $V_p = 60$) over the designated time horizon, then the unexpected loss is equal to $39. We can conclude that the VaR 99% is $39$, and there is a corresponding 1% probability that losses will exceed the VaR of $39$ over the measurement period. If the bank wanted to hold economic capital such as to achieve a target credit rating on its senior-rated debt equal to a 1% probability of default, it would need to set aside $39 in economic capital to support the loan (excluding any diversification contribution of the loan on the existing portfolio).

It is worth noting that VaR based on an expected value benchmark is unaffected by a constant shift in the entire distribution of returns. In the above example, an economic recession may cause expected losses to rise, but not the VaR. In this sense, the efficacy of the VaR measure is reduced because it is less sensitive to weak economic conditions which would induce a decline in returns under all states of nature – there is no change in the magnitude of the risk measure despite a larger absolute loss at the given confidence interval.

The preceding example shows how the determination of economic capital in a bank is consistent with the VaR concept of estimating the distance between expected and unexpected outcome. The VaR confidence level is scaled to the critical threshold level for determining the amount of economic capital deemed necessary to protect the bank against adverse events. In the desire that banks monitor and manage the size of lower-tail outcomes so that the probability of financial distress is low, regulators have adopted VaR as the standard for measuring risk for determining bank capital adequacy. Szegö (2002) notes that the second consultative paper of the new Basel Accord assumes the VaR concept as the risk measure for deriving minimum capital standards, and requires in its solution that the risk of each loan must be portfolio invariant. Further, the Accord requires that regulatory capital for each loan must be correlated to its marginal contribution to the VaR.\footnote{Szegö (2002), p.1258.} \footnote{Ibid, p.1259.}
4.3. Expected Shortfall

Expected shortfall is defined as the conditional expectation of loss given that the loss is beyond the VaR level. Thus, by definition, expected shortfall measures losses beyond the VaR level. In terms of our earlier criteria, expected shortfall corresponds to a Type 2 risk measure.

If \( x \) is the profit/loss of a portfolio \( X \), with positive values of \( x \) representing profits and negative values (\(-x\)) representing losses, and \( VaR_\alpha(X) \) is the VaR at the \( \alpha \) percent confidence level, expected shortfall at the \( \alpha \) percent confidence level \( (ES_\alpha) \) is defined as follows:

\[
ES_\alpha(X) = VaR_\alpha(X) + (1 - \alpha)^{-1} E \{\max \{-x - VaR_\alpha(X), 0\}\},
\]

where \( E(\cdot) \) is the expected value operator.

This definition shows that \( ES_\alpha(X) \) is more sensitive to the severity of losses exceeded \( VaR_\alpha(X) \) given expected shortfall is calculated by taking the expected value of all losses which are greater than or equal to \( VaR_\alpha(X) \). If investors (and regulators) are concerned not with the potential loss that would occur at a specified confidence level, but rather, the severity of losses beyond the VaR level, the expected shortfall may be considered to be a more suitable measure of risk than the VaR.

4.4. First-Order Lower Partial Moment

As defined earlier, lower partial moments measure risk in terms of deviations below a loss threshold or target rate of return. Earlier the general class of lower partial moments, for a continuous distribution, was represented as follows:

\[17\] Other names for expected shortfall in the literature include tail conditional expectation, tail VaR, conditional VaR and conditional loss.
The first-order lower partial moment (LPM₁) has a power of \( n = 1 \), and thus measures the weighted average deviation from the target level. LPM₁ is related to the expected shortfall risk measure in that it provides the expected loss relative to the loss threshold or benchmark return \((t)\). Appendix 2 provides an example to show that LPM₁ and expected shortfall provide an identical risk measure \( \rho(X) \) when the losses are measured in terms of the same loss \((\alpha)\) quantile. Like expected shortfall, LPM₁ corresponds to a Type 2 risk measure in terms of our earlier defined criteria.

While LPM₁ and expected shortfall produce the same measure of risk for losses beyond the loss threshold, the LPM₁ risk measure has a significant advantage over expected shortfall in that while expected shortfall (like VaR) is usually measured in terms of a specific loss quantile, the LPM₁ can be calculated on the basis of deviations from zero \((t = 0)\). This enables the full distribution of losses to be taken into consideration in the risk measure, rather than expected losses beyond the loss threshold. If investors are concerned with all losses (or below target returns), rather than those that are greater than the loss threshold, then the lower partial moment class of risk measures, with \( t = 0 \), is a more complete measure of risk.

It is worth highlighting the implications of the above from the perspective of external regulatory risk measures. If regulators are concerned only with protection against bankruptcy, then a VaR measure may be appropriate. If regulators, however, are concerned that a bank be sufficiently capitalised to cover the size of losses in the event of bankruptcy (losses beyond the predetermined loss threshold), then expected shortfall is a more relevant risk measure for regulatory purposes. But what of smaller losses below the loss threshold, being losses that occur with a less than \((\alpha)\) confidence interval? A regulatory risk measure based on VaR or expected shortfall implicitly assumes that losses that are less than the threshold are self-insured by the bank or that the bank can efficiently recapitalise in the event that it needs to raise equity to cover these unexpected losses. If, however, a bank frequently suffers losses less than the
threshold, it may find insurance or recapitalisation costly, particularly in times of
economic slowdown where loan losses are likely to be larger and equity
recapitalisation more expensive.

The lower partial moment class of downside risk measures incorporate all losses into
the measure, both above and below the predetermined confidence interval, when the
target is set at \((t = 0)\). Risk measures that focus either on the probability of losses
(VaR) or extreme losses (expected shortfall) fail to capture the likely systemic impact
of banks incurring frequent losses below the predetermined loss threshold. If
regulators determine that bank capital requirements should be based on either of the
above measures of unexpected losses, then they are ignoring the potential for loss in
confidence in the banking system if banks do incur frequent unexpected losses below
the threshold and subsequently find it difficult to recapitalise. In these circumstances,
LPM\(_1\) (or a higher order LPM risk measure) is likely to be a more appropriate than
VaR or expected shortfall for determining bank capital requirements.

4.5. Second-Order Lower Partial Moment

With respect to the general class of lower partial moment risk measures, as the power
function \(n\) increases, larger deviations from the threshold are penalised more than
smaller deviations. The second-order lower partial moment (LPM\(_2\)) places a quadratic
penalty on deviations below the threshold \((n = 2)\). Formally, LPM\(_2\) represents the
semi-variance or lower partial variance (Markowitz, 1959), and the square root of the
lower partial variance represents the downside standard deviation. By placing a larger
penalty on larger losses, LPM\(_2\) corresponds to a Type 3 risk measure in terms of our
earlier defined criteria.

Like LPM\(_1\), when the target level from which deviations are measured is set to cover
all losses \((t = 0)\), LPM\(_2\) captures in the risk measure smaller unexpected losses that are
not included in measures based on loss thresholds linked to predetermined confidence
intervals (such as VaR and expected shortfall). This means that unlike VaR and
expected shortfall, the LPM₂ risk measure does not create an incentive for portfolio managers to take actions that increase the cumulative distribution function for losses that are smaller than the loss threshold. Such actions would increase the risk of the portfolio but not be captured in risk measures that base losses relative to a loss threshold. Managers may be motivated to take such actions to increase the risk-adjusted return on their portfolios and increase their remuneration where bonuses are linked to such measures. Risk measures that are based on the full distribution of losses should not entice such gaming on the part of managers.

The use of the semi-variance as a basis for portfolio optimisation (asset allocation) decisions has been the subject of ongoing research. Markowitz (1959) argues that the semi-variance as a risk measure tends to produce more efficient portfolios than portfolios based on variance as the risk measure. Further, semi-variance can be calculated relative to an investor-specific benchmark. For a summary of empirical research on the use of semi-variance for asset allocation, see Narwocki (1992).

4.6. Findings

Any risk measure that is based on the general class of lower partial moments is a suitable candidate for an incentive-compatible risk measure because of the mathematical relationship between lower partial moments of order \( n \) and stochastic dominance criteria. The five risk measures reviewed above have been selected because they are derivations of the general class of lower partial moments. By analysing the risk measures in terms of lower partial moments we can determine the risk attitude implicit in each measure.

Shortfall probability is measured by the zero order lower partial moment (LPM₀) and is consistent with a risk-seeking utility function. VaR can be interpreted as a special case of shortfall probability – fixing the probability of the LPM₀ gives the corresponding VaR measure. VaR is a suitable risk measure for an investor having a positive first derivative (non-satiety), but risk-aversion on the part of the investor is not a necessary condition. Expected shortfall and LPM₁ produce the same measure of
risk for losses when the target threshold is identical. They only differ in that expected shortfall typically corresponds to a α percent confidence level while LPM₁ can be based on negative deviations from any target level. The power function of one indicates that these measures are consistent with a risk-neutral utility function. The quadratic power function of LPM₂ means large deviations from the threshold are penalised more than smaller deviations in the risk measure, consistent with a risk-averse attitude to losses.

5. COHERENCY OF RISK MEASURES

5.1. Axioms of coherence

Artzner et al (1999) present and justify a set of four desirable structural properties for measures of risk, which they argue should hold for any risk measure which is to be used to effectively regulate or manage risks. They call measures that satisfy these properties ‘coherent’. In this section we examine the properties of coherent risk measures, and determine if coherency is a desirable characteristic for risk measures used within the bank where the aim is to create incentive compatibility between the risk preferences of the centre of the bank and the investment decisions of credit portfolio managers.

Let X and Y represent the random outcomes of two risky assets and ρ(X) and ρ(Y) represent risk measures for these investments. Further, rf represents the risk-free rate of return, and α and λ are positive numbers. The four axioms that characterise coherent risk measures, translation invariance, monotonicity, positive homogeneity and subadditivity, are represented as follows:

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Property</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axiom 1</td>
<td>Translation Invariance</td>
<td>$\rho(X + \alpha rf) = \rho(X) - \alpha$</td>
</tr>
<tr>
<td>Axiom 2</td>
<td>Monotonicity</td>
<td>$\rho(X) \geq \rho(Y)$ if $X \leq Y$</td>
</tr>
<tr>
<td>Axiom 3</td>
<td>Positive Homogeneity</td>
<td>$\rho(\lambda X) = \lambda \rho(X)$</td>
</tr>
<tr>
<td>Axiom 4</td>
<td>Subadditivity</td>
<td>$\rho(X + Y) \leq \rho(X) + \rho(Y)$</td>
</tr>
</tbody>
</table>

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We discuss each in turn, and their implications for the opportunity set of incentive-compatible risk measures.

5.2. Axiom 1: Translation Invariance

Artzner et al (1999) define risk in terms of the variability of the future value of a position due to market changes or uncertain events. In determining the desirability of a risk measure, they focus on the random variables on the set of states of nature at a future date, interpreted as possible future values of positions or portfolios currently held.\(^{18}\)

Consider an investor who desires fixed future wealth of \(Z\). In order to achieve this objective, the investor can invest in either a risky asset \(X\), or a combination of the risky asset and an amount \(\alpha\) in a non-risky investment that guarantees a certain outcome equal to \((\alpha r_f)\). The translation invariance axiom implies that investment in the risk-free asset today reduces the amount that needs to be invested in the risky asset where the objective is to achieve a certain future portfolio value. As the risk-free investment provides a guaranteed profit, it reduces the potential losses arising from investment in the risky asset by exactly the amount invested in the risk-free asset:

\[
\rho(X + \alpha r_f) = \rho(X) - \alpha
\]

In terms of capital requirements, the axiom indicates that if a risk-free investment of \(\alpha\) is added to a risky portfolio, then the capital requirement should decrease by the amount \(\alpha\). In the extreme case where the amount invested in the riskless asset is set such that \(\alpha = \rho(X)\), then \(\rho(X + \rho(X) r_f) = 0\). This justifies a capital requirement equal to \(\rho(X)\) to cover the risk of loss, rendering the position in asset \(X\) acceptable without further capital injection.

The risk measures examined in this study that characterise risk as the overall seriousness of potential losses (VaR and ES) are translation invariant. If a risk-free asset is added to a risky portfolio and the profit and loss distribution reconstructed to

reflect the addition, each profit and loss value will be reduced by the amount added. Consequently both the VaR and the ES for the portfolio, at the designated loss threshold, will be lower in accordance with the translation invariance axiom.\textsuperscript{19} However, for those risk measures examined in this study that represent risk as the magnitude of deviations from a prespecified target (shortfall probability, LPM\textsubscript{1} and LPM\textsubscript{2}), translation invariance in terms of the risk-free condition of Artzner et al (1999) is not achieved.\textsuperscript{20} For the class of LPM\textsubscript{n} risk measures, the requirement that the risk measure value decrease by the amount of the investment in the risk-free asset will hold only if the cash flow from the risk-free asset is matched by the value of the worst loss of the risky asset.

The failure of our recommended incentive compatible risk measure, LPM\textsubscript{2}, to conform to the translation invariance axiom is not of consequence. Barbosa and Ferreira (2004) claim that the translation invariance axiom is too restrictive in the sense that it is not necessary that the risk measure value decrease by the exact amount of the investment in risk-free asset.\textsuperscript{21} If a risk measure is decreased (increased) when a position in a risk-free asset is added (withdrawn) from a risky portfolio, then the risk measure motivates investment in risk-free assets, regardless of whether the measure declines by the exact amount of the risk-free asset.

It is also worth noting that a number of other authors place less restrictive conditions on the property of translation invariance. In a similar vein to Artzner et al (1999), Pedersen and Satchell (1998) specify axioms that represent desirable properties of a financial risk measure. Their axioms cover non-negativity, homogeneity, subadditivity and shift-invariance. The shift-invariance axiom of Pedersen and Satchell (1998) differs from translation invariance axiom of Artzner et al (1999) in that the former makes the risk measure invariant to the addition of a constant to the random variable:

\[ \rho(X + \lambda) \leq \rho(X) \quad \text{for all real } \lambda. \]

\textsuperscript{19} See Tasche (2002) for proof that VaR is translation invariant and Acerbi and Tasche (2002a, 2002b) for proof that ES is coherent.
\textsuperscript{20} See Barbosa and Ferreira (2004) p.25 for a proof that LPM\textsubscript{n} does not satisfy the translation invariance axiom.
\textsuperscript{21} Barbosa and Ferreira (2004), p.6.
This indicates that the risk measure may be either unchanged or decreased by the addition of a constant, and is thus less restrictive than the translation invariance of Artzner et al (1999). Gaivoronski and Pflug (2001) specify the translation invariance condition for a risk measure as follows:

$$\rho(X + t) = \rho(X) \text{ for all real } t$$

In this interpretation of the translation invariance condition, the risk of a portfolio cannot be changed by adding to the portfolio a fixed sum of riskless money. This is in keeping with the general results for the LPM categories of measures, where adding a constant to a random variable results in a new random variable with the same deviation around the mean.

### 5.3. Axiom 2: Monotonicity

The monotonicity axiom says that if a portfolio X is always worth less than a portfolio Y in terms of all possible outcomes, then the risk measure of X should be greater than the risk measure of Y. From an economic perspective, the axiom implies that portfolios embodying higher potential losses should report a larger risk measure and require more risk capital.

The shortfall probability measure does not conform to the monotonicity axiom because it assigns a larger risk measure to the least risky portfolio, reflecting the convex attitude to risk implicit in the measure. VaR, ES and LPM$_1$ do conform to the monotonicity axiom, as proven by various authors. Szego (2002) asserts that monotonicity rules out any semi-variance type of risk measure, where we note that LPM$_2$ represents the semi-variance when losses are measured as deviations below the expected value of the portfolio. In contrast, Barbosa and Ferreira (2004) claim the measure LPM$_{n,t}$ satisfies monotonicity when $n > 0$. This clearly captures LPM$_2$. We examine this conflict below.

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The monotonicity axiom implies that returns can be used to determine the risk ranking of instruments or portfolios. Indeed, monotonicity suggests that if one portfolio Y has greater losses than another portfolio X, such that the probability of observing an outcome below any threshold is lower for portfolio X, then portfolio X should stochastically dominate portfolio Y.

The problem that arises is risk measures that are coherent in terms of Artzner et al (1999) are generally not consistent with first or second-order stochastic dominance principles, and are never likely to be consistent with third-order stochastic dominance principles. This was shown earlier, where only the LPM₂ risk measure provided a risk-ranking of portfolios that was consistent with the ranking of the portfolios by both SSD and TSD. In particular, we showed that the VaR, ES and LPM₁ measures were not increasing with the risks of the portfolios, and indeed, in some cases were lower as the risk of the portfolios increased.²⁴ Yet these measures are monotonic increasing in terms of Artzner et al (1999), and ES and LPM₁ are deemed coherent risk measures.²⁵

The inconsistency of coherent risk measures with TSD arises because TSD dominance of one risky portfolio over another implies dominance in the third-distribution function, which as shown implies the expectation of squared profits and losses at each point in the distribution. More specifically, we showed earlier that if an investor exhibits non-satiety, risk aversion and a preference for positive skewness in the distribution of returns, then \( \rho(X_1) < \rho(X_2) \) if the following holds:

\[
\int_{-\infty}^{t} (t-x)^{(n-1)} f_1(x) \, dx \leq \int_{-\infty}^{t} (t-x)^{(n-1)} f_2(x) \, dx \quad \text{for all} \, x, x \leq t,
\]

where \( f_1(x) \) and \( f_2(x) \) are the density functions of \( X_1 \) and \( X_2 \) and \( n > 2 \). Recall that this condition employs the lower partial moment of degree \( (n-1) \) to penalise large

²⁴ We ignore shortfall probability from this discussion given the underlying risk attitude in this measure is one of risk-seeking.
deviations from the loss threshold more than smaller deviations from the loss threshold. At \( n = 3 \) there is a quadratic penalty on deviations below the loss threshold, which was shown to be consistent with portfolio \( X_1 \) dominating portfolio \( X_2 \) according to TSD. This is inconsistent with coherent risk measures, which do not place a larger penalty on larger deviations from the target threshold.\(^{26}\)

The lack of consistency between coherent risk measures and stochastic dominance principles implies that if we restricted our internal risk measures to only those measures that are coherent, then LPM\(_2\) would be omitted from our list of candidates and we would have no risk measure that is concurrently coherent and incentive-compatible with the risk preference function of the centre. If we are restricted to the condition that risk measures are coherent in terms of Artzner et al (1999), then the set of acceptable risk measures would allow managers to select portfolios that are dominated by TSD. This contradicts the risk-preference function of the centre. For this reason, we suggest that the monotonicity axiom be replaced by the stronger condition that the risk measure provides a risk-ranking that is consistent with TSD principles, where the axioms are to be applied to risk measures for use within the banking firm.\(^{27}\)

### 5.4. Axiom 3: Positive Homogeneity

The positive homogeneity axiom indicates that if an investor purchases the same risk twice (identical portfolios), then the risk should be doubled. This suggests the risk measure should not be influenced by the size of the position, and for all \( \lambda \geq 0 \), the risk is scalar multiplicative.\(^{28}\) For positive homogeneity to hold there should be no diversification effect across portfolios with identical payoff distributions.

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\(^{26}\) As pointed out by Barbosa and Ferreira (2004) p.22, coherent risk measures involve the expectation of non-squared profit and losses.

\(^{27}\) This is in keeping with De Giorgi (2005), who includes SSD as a property that reward and risk measures should satisfy for portfolio selection.

\(^{28}\) It holds that the liquidity of the position should not be influenced by the size of the position. If this is the case, positive homogeneity will not hold.
The shortfall probability measure is not positive homogenous. The probability of loss of two combined portfolios that are identical is exactly the same as the probability of loss of each of the individual portfolios. Shortfall probability thus does not exhibit scalar multiplicativity. Supremo (2001) shows that VaR and ES are positive homogenous – the VaR and ES for identical portfolios is doubled when the portfolios are summed. Similarly, the LPM\(_1\) for identical portfolios that are combined into one portfolio is equal to the sum of the LPM\(_1\) for the individual portfolios. LPM\(_2\), however, is not positive homogenous owing to the quadratic power function – the LPM\(_2\) measure for two identical combined portfolios will always exceed the sum of the LPM\(_2\) measures for each of the individual portfolios. This will hold for any LPM\(_n\) measure where losses are measured as deviations from the expected value and for which \(n > 1\). Further, the LPM\(_n\) measure for two or more combined identical portfolios will be lower than the sum of the LPM\(_n\) measures for each of the individual portfolios when \(0 < n < 1\). It can be shown that shortfall probability and LPM (where \(0 < n < 1\) and \(n > 1\)) fail to meet the positive homogeneity axiom. It can also be shown that LPM\(_1\) is positive homogenous.

The significance of these results is that the incentive-compatible risk measure LPM\(_2\) fails to meet the positive homogeneity axiom and is thus not a coherent risk measure in terms of Artzner et al (1999). While we have found that the failure of LPM\(_2\) to meet the translation invariance axiom is not of major consequence and that the monotonicity axiom should be replaced with the stronger condition of congruence with TSD, we must assess if the failure of LPM\(_2\) to meet the positive homogeneity axiom is to its detriment for use as an incentive-compatible risk measure within the bank.

Whether or not the failure of LPM\(_2\) to meet the positive homogeneity axiom is of consequence depends to some extent on how the performance of credit portfolios is measured within the bank. If performance is measured on the basis of individual loans, then the fact that LPM\(_2\) is not positive homogenous should be of little concern because a credit portfolio manager will be judged on the basis of the sum of the individual loans that make up the portfolio under the control of the manager. This
would not seem an unrealistic assumption subject to the extent to which loans are priced and managed on an individual basis. If, however, the performance of a credit portfolio manager is based on the aggregated portfolio of loans under the control of the manager, meaning risk measures are based on the portfolio rather than the sum of the individual loans in the portfolio, then the use of LPM₂ would overstate the risk of the portfolio when the loans carry no diversification benefits.²⁹ In this case, the failure of LPM₂ to meet the positive homogeneity axiom is of significance. If the risk measure overstates the risk of the portfolio when loans of identical risk are added to the portfolio, the credit portfolio manager will have a greater propensity to reject loans where the risk of the portfolio will be significantly (and incorrectly) overstated. This means loans that are valuable to the bank may be rejected.

Some may take the view that it is appropriate for a credit portfolio manager to resist adding loans to a portfolio where the risks are positively correlated, and in this regard, the penalty placed on the portfolio when using LPM₂ serves to make this risk measure attractive. We argue that the opposite is the case. Many credit portfolio managers in large banks will specialise in a certain loan type, region or industry. Specialisation such as this offers information advantages and other cost economies, and is desirable. Under these conditions, the credit manager will face very few opportunities where loans can be written that provide significant diversification benefits. Indeed, it is likely that most loans will be positive correlated in terms of the distribution of returns.

Diversification across loan portfolios is more likely to be a higher-level function within the bank because it is at the head office or business unit level that diversification strategies are determined and diversification opportunities more easily identifiable. Thus accountability for diversification across loan portfolios or regions rests at a higher level within the bank than that of the line manager – unless there are opportunities to add loans with uncorrelated loss distributions to the portfolio.³⁰ In the case where loans of identical loss distributions are added to a portfolio, and where the

²⁹ The same result holds for any LPM measure where \( n > 1 \).

³⁰ This suggests that skilled managers should be better able to identify potential diversification benefits among the opportunity set of loans available to them. Such managers should be rewarded for these skills through the risk measure.
performance of managers is measured on a portfolio basis, we argue that the risk of the portfolio should be scalar multiplicative and positive homogeneity must hold. If a manager purchases the same risk twice, the risk of the portfolio should be doubled. This will not be the case if LPM$_2$ is used as the basis for measuring risk within the bank.$^{31}$ If, however, the performance of managers is measured on an individual loan basis, then the failure of LPM$_2$ to meet positive homogeneity is of little consequence.

Prior to concluding this section, it is worth noting that some authors believe that the positive homogeneity assumption that risk increases proportionally to the initial wealth placed on a position does not reflect investors’ perceptions of risk. De Giorgi (2005) quotes cases of laboratory experiments that suggest decision-makers become more risk averse with a larger net payoff (positive and negative).$^{32}$ In this case it would not be appropriate to impose that a risk measure satisfies the property of positive homogeneity.

The decision to measure the performance of managers on a portfolio or individual loan basis depends, to a large extent, on the degree to which the manager faces opportunities to reduce the risk of the portfolios under their responsibility through diversification. It is with this in mind that we assess the final axiom for a coherent risk measure: subadditivity.

5.5. Axiom 4: Subadditivity

A risk measure $\rho$ is said to be subadditive when the risk of the combined position of two investments, $X$ and $Y$, is less or equal to the sum of the risk of the individual portfolios:

$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$

$^{31}$ Note that a risk-seeking manager would have a preference to use a LPM risk measure of order $0 \leq n < 1$ because the risk of the portfolio is understated when loans of identical risk are added to a portfolio. The risk of the portfolio in these cases is less than the sum of the individual loans. Given there are no diversification benefits when loans of identical risk are combined in a portfolio, the LPM measure with $0 \leq n < 1$ sends an incorrect signal regarding the true underlying risk of the portfolio.

Subadditivity embodies the notion that portfolio diversification results in a reduction in risk when there is less than perfect positive correlation in the returns in the individual investments that comprise the portfolio. The subadditivity axiom implies that the act of combining uncorrelated risks in a portfolio should never increase the risk measure or the capital requirement.

Artzner et al (1999) contend that subadditivity is a natural requirement for a risk measure for a number of reasons:

1. If a risk measure fails to incorporate diversification benefits then an individual will have an incentive to establish two separate trading accounts, one for each risk, in order to lower the overall margin requirement. In this vein, a credit portfolio manager within a bank could act to lower the apparent credit risk of a portfolio (and any subsequent capital requirements) by artificially splitting the portfolio into smaller holdings or individual credits.

2. A banking institution may have an incentive to break up into various subsidiaries in order to reduce the overall regulatory capital requirement if the risk measure used for determining minimum capital requirements does not reward diversification benefits. In this case the non-subadditive risk measure encourages regulatory capital arbitrage.

3. At the business unit level, a bank can allocate capital among managers or trading desks in the knowledge that the global risk for the unit is less than the sum of local risks at the line level. Subadditivity of the selected risk measure ensures risk management can be decentralised in this way.

Some authors argue that these arguments do not hold for certain types of risk.33 For example, consider the case of two catastrophe bonds for which the risks are independent. One bond is linked to earthquake in City A and the other linked to earthquake in City B, and both cities are in different geographic regions. If an

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earthquake occurs, the payment of interest and principal to the holder of the bond is reduced or eliminated. Should an investor place their available investment funds into one bond, or diversify and put equal amounts into two bonds? The subadditivity property would suggest that investment in two bonds is more appropriate because the likelihood of earthquakes occurring in both cities at the same time is highly remote. However, if the investor is concerned with the probability of default, then a portfolio comprising investment in both bonds will have a higher default probability than an investment in a single bond. This arises because the probability that an earthquake occurs in at least one city is larger than the probability that earthquake occurs in either city.\textsuperscript{34} Such an example suggests that the relevance of coherency axioms depends on which characteristics of risk are relevant to those responsible for managing risk. While the probability of default may be the element of risk of most interest for some investors, we have established that the aspect of risk most relevant to bank stakeholders is the size of loss in the event of default.

In the case of the banking firm, subadditivity considerations are relevant for the determining the relevant internal measure of risk. If a credit portfolio manager identifies an opportunity to add a loan to the portfolio that has diversification benefits in losses, then it is in the interests of the centre of the bank that such loans are obtained. If, however, the risk measure used to assess the performance of the portfolio manager is not subadditive, then there will be cases where it is against the interests of the credit manager to add the loan to the portfolio, despite the real underlying benefits to bank stakeholders. In terms of the subadditivity axiom, this will be the case where the risk measure for the diversified portfolio is larger than the sum of the individual credits that make up the portfolio. If credit portfolio managers are to be encouraged to use or develop their skills to identify loans that provide diversification benefits to the bank, then for incentive-compatibility, it is a requirement that the risk measure is subadditive.

\textsuperscript{34} If the probability of earthquake in City A is 1% and the probability of earthquake in City B is 1%, then an investment in one bond will have a 1% probability of default, while alternatively an investment in two bonds will have a default probability of approximately 2%.
If the performance of credit portfolio managers is measured on the basis of individual loans that make up their portfolios, then there will be little incentive for managers to identify loans that provide diversification benefits for the bank. This is because the risk measure used for performance measurement will provide no recognition or reward for identifying individual credits that provide such benefits. It is for this reason that we argue that the performance of credit portfolio managers should be assessed on a portfolio basis. Incentive-compatibility considerations require that managers add loans to their portfolios that are optimal from the perspective of the centre of the bank and the stakeholders that it represents.

Which risk measures from the set of candidates are subadditive?

In order to address this question, we examine three hypothetical loan portfolio distributions that combine two individual loans possessing diversification benefits in the domain of losses. These portfolios are presented in Tables 1, 2 and 3. Each table shows the value of the individual loans under various states of nature and the probability distribution for each loan. The tables also show the risk measures for the individual loans, the sum of the risk measures for the individual loans, and the risk measures for the portfolio that combines the individual loans. In terms of Artzner et al (1999), subadditivity holds if the risk measure for the portfolio is less than or equal to the sum of the risk measures for the individual loans.

Table 1 considers two loans, X and Y. The loans have a face value of $100, an expected value of $98 and are diversified in the domain of losses. There are five states of nature. Under the first state of nature, loan X has a value of $50 and loan Y has a value of $90, both with equal probability of occurrence. Under the second state of nature, X has a value of $90 and Y a value of $100, while X has a value of $90 and Y a value of $50 under the third state. Under the fourth state, X has a value of $100 and Y a value $90. Under the fifth state both loans have a value of $100. Given these values under each state of nature, it is clear that the loans are diversified in the domain of losses (where losses are measured as shortfalls below the expected value). Risk measure calculations for the individual loans and the portfolio comprising the loans
are presented at the bottom of the table. The risk measures are VaR and ES at the 95% confidence level, LPM of orders n = 0, 0.5, 1 and 2,\textsuperscript{35} the downside semi-deviation (DSD)\textsuperscript{36} and the Wang Transform at the 95% confidence level.

The results in Table 1 show that both VaR and LPM\textsubscript{2} fail subadditivity because for these measures, the value of the risk measure for the portfolio exceeds the sum of the risk measures for the individual loans. If the performance of credit managers within the bank was to be assessed on a portfolio basis, managers would in this case not add loan Y to a portfolio with a payoff replicating loan X because the risk measure would for the portfolio overstates the true risk. These measures place a penalty on diversification, which is clearly not optimal from the perspective of the centre of the bank. Of particular concern is the finding that the only measure that we have found to be incentive-compatible with the centre of the bank in terms of expected utility, LPM\textsubscript{2}, is not subadditive. If the use of LPM\textsubscript{2} acts to discourage managers from seeking and adding loans into their portfolios that provide risk-reducing benefits, then the risk measure must fail our overall test of incentive-compatibility between managers and the centre of the bank.

\textsuperscript{35} For LPM calculations, losses are measured as deviations below the expected loan value. This is consistent with earlier calculations in the chapter.
\textsuperscript{36} The downside semi-deviation (DSD) is the square root of the semi-variance (LPM\textsubscript{2}).
Table 1
Subadditivity for Loans X and Y: Risk Measures

<table>
<thead>
<tr>
<th>Loan X</th>
<th>State</th>
<th>X</th>
<th>P(X)</th>
<th>E(X)</th>
<th>Losses</th>
<th>Weighted Tail</th>
<th>$\sqrt{\text{E}(X)-X}$ P(X)</th>
<th>$\text{E}(X)-X$ P(X)</th>
<th>$\text{E}(X)-X^2$ P(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>50</td>
<td>3%</td>
<td>98.0</td>
<td>48.0</td>
<td>30.0</td>
<td>0.208</td>
<td>1.440</td>
<td>69.120</td>
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<tr>
<td></td>
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<td>98.0</td>
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<td>0.057</td>
<td>0.160</td>
<td>1.280</td>
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<tr>
<td></td>
<td>3</td>
<td>90</td>
<td>3%</td>
<td>98.0</td>
<td>8.0</td>
<td>36.0</td>
<td>0.085</td>
<td>0.240</td>
<td>1.920</td>
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<td></td>
<td>4</td>
<td>100</td>
<td>2%</td>
<td>98.0</td>
<td>8.0</td>
<td>36.0</td>
<td>0.085</td>
<td>0.240</td>
<td>1.920</td>
</tr>
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<td></td>
<td>5</td>
<td>100</td>
<td>90%</td>
<td>98.0</td>
<td>8.0</td>
<td>36.0</td>
<td>0.085</td>
<td>0.240</td>
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<td></td>
<td>Sum</td>
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<td></td>
<td>64.0</td>
<td>66.0</td>
<td>0.349</td>
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<table>
<thead>
<tr>
<th>Loan Y</th>
<th>State</th>
<th>X</th>
<th>P(X)</th>
<th>E(X)</th>
<th>Losses</th>
<th>Weighted Tail</th>
<th>$\sqrt{\text{E}(X)-X}$ P(X)</th>
<th>$\text{E}(X)-X$ P(X)</th>
<th>$\text{E}(X)-X^2$ P(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>50</td>
<td>3%</td>
<td>98.0</td>
<td>48.0</td>
<td>30.0</td>
<td>0.208</td>
<td>1.440</td>
<td>69.120</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>90</td>
<td>2%</td>
<td>98.0</td>
<td>8.0</td>
<td>36.0</td>
<td>0.057</td>
<td>0.160</td>
<td>1.280</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>90</td>
<td>3%</td>
<td>98.0</td>
<td>8.0</td>
<td>36.0</td>
<td>0.085</td>
<td>0.240</td>
<td>1.920</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>100</td>
<td>2%</td>
<td>98.0</td>
<td>8.0</td>
<td>36.0</td>
<td>0.085</td>
<td>0.240</td>
<td>1.920</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>100</td>
<td>90%</td>
<td>98.0</td>
<td>8.0</td>
<td>36.0</td>
<td>0.085</td>
<td>0.240</td>
<td>1.920</td>
</tr>
<tr>
<td></td>
<td>Sum</td>
<td>100%</td>
<td></td>
<td>64.0</td>
<td>66.0</td>
<td>0.349</td>
<td>1.840</td>
<td>72.320</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio (X+Y)</th>
<th>State</th>
<th>X</th>
<th>P(X)</th>
<th>E(X)</th>
<th>Losses</th>
<th>Weighted Tail</th>
<th>$\sqrt{\text{E}(X)-X}$ P(X)</th>
<th>$\text{E}(X)-X$ P(X)</th>
<th>$\text{E}(X)-X^2$ P(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>140</td>
<td>3%</td>
<td>196.0</td>
<td>56.0</td>
<td>70.0</td>
<td>0.224</td>
<td>1.680</td>
<td>94.080</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>140</td>
<td>3%</td>
<td>196.0</td>
<td>56.0</td>
<td>70.0</td>
<td>0.224</td>
<td>1.680</td>
<td>94.080</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>190</td>
<td>2%</td>
<td>196.0</td>
<td>6.0</td>
<td>36.0</td>
<td>0.049</td>
<td>0.120</td>
<td>0.720</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>190</td>
<td>2%</td>
<td>196.0</td>
<td>6.0</td>
<td>36.0</td>
<td>0.049</td>
<td>0.120</td>
<td>0.720</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>200</td>
<td>90%</td>
<td>196.0</td>
<td>6.0</td>
<td>36.0</td>
<td>0.049</td>
<td>0.120</td>
<td>0.720</td>
</tr>
<tr>
<td></td>
<td>Sum</td>
<td>100%</td>
<td></td>
<td>124.0</td>
<td>140.0</td>
<td>0.547</td>
<td>3.600</td>
<td>189.600</td>
<td></td>
</tr>
</tbody>
</table>

Risk Measures

<table>
<thead>
<tr>
<th>Risk measure</th>
<th>Investment X</th>
<th>Investment Y</th>
<th>Sum X, Y</th>
<th>Portfolio</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR 95%</td>
<td>8.000</td>
<td>8.000</td>
<td>16.000</td>
<td>56.000</td>
<td>Fails subadditivity</td>
</tr>
<tr>
<td>ES 95%</td>
<td>32.000</td>
<td>32.000</td>
<td>64.000</td>
<td>56.000</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM0</td>
<td>0.080</td>
<td>0.080</td>
<td>0.160</td>
<td>0.100</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM0.5</td>
<td>0.349</td>
<td>0.349</td>
<td>0.699</td>
<td>0.547</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM1</td>
<td>1.840</td>
<td>1.840</td>
<td>3.680</td>
<td>3.600</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM2</td>
<td>72.320</td>
<td>72.320</td>
<td>144.640</td>
<td>189.600</td>
<td>Fails subadditivity</td>
</tr>
</tbody>
</table>
Now consider Table 2, which shows risk measures for two individual loans, F and G, and the portfolio that combines these loans. These loans also have a face value of $100, but their expected value of $98.3 is higher than the previous case. These loans also have lower volatility than for the previous case, but also offer similar diversification benefits in the domain of losses. In the first state of nature, loan F has a value of $50 while loan G has a value of $100. In the second state, loan F has a value of $90 while loan G has a value $100, and in the third state loan F has a value of $100 while loan G drops to $50. In the fourth state, F has a value of $100 and G has a value of $90, and both loans have a value of $100 in the fifth state.

In the case of loans F and G, only the VaR risk measure fails subadditivity, with the VaR of the portfolio exceeding the sum of the VaR of the individual loans. The shortfall probability measure (LPM0) is weakly subadditive, in the sense that the risk measure for the portfolio matches the sum of the risk measures for the individual loans. While this risk measure does not penalise portfolio diversification, it is notable that the measure fails to reward diversification. If a risk measure does not reward diversification, this may act as a disincentive to credit portfolio managers to actively seek loans with diversification benefits, and at the very least, make them indifferent about adding such loans to a portfolio. We assert that such indifference is not congruent with the risk objectives of the centre of the bank.

It is worth noting that the LPM2 risk measure does reward diversification in the case of loans F and G, with the value of the risk measure for the portfolio being less than the sum of the value of the risk measures for the individual loans. This indicates that for certain portfolio distributions, the measure may encourage diversification. Its failure, however, to consistently recognise diversification benefits rules it out as an incentive-compatible risk measure in our bank setting.
### Table 2
Subadditivity for Loans F and G: Risk Measures

#### Loan F

<table>
<thead>
<tr>
<th>State</th>
<th>X</th>
<th>P(X)</th>
<th>E(X)</th>
<th>Losses</th>
<th>Weighted Tail</th>
<th>([E(X)-X]^{0.5} P(X))</th>
<th>([E(X)-X] P(X))</th>
<th>([E(X)-X]^2 P(X))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
<td>3%</td>
<td>98.3</td>
<td>48.3</td>
<td>30.0</td>
<td>0.208</td>
<td>1.449</td>
<td>69.987</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>2%</td>
<td>98.3</td>
<td>8.3</td>
<td>36.0</td>
<td>0.058</td>
<td>0.166</td>
<td>1.378</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>3%</td>
<td>98.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>2%</td>
<td>98.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>90%</td>
<td>98.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td>56.6</td>
<td>66.0</td>
<td>0.266</td>
<td>1.615</td>
<td>71.364</td>
</tr>
</tbody>
</table>

#### Loan G

<table>
<thead>
<tr>
<th>State</th>
<th>X</th>
<th>P(X)</th>
<th>E(X)</th>
<th>Losses</th>
<th>Weighted Tail</th>
<th>([E(X)-X]^{0.5} P(X))</th>
<th>([E(X)-X] P(X))</th>
<th>([E(X)-X]^2 P(X))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>50</td>
<td>3%</td>
<td>98.3</td>
<td>48.3</td>
<td>30.0</td>
<td>0.208</td>
<td>1.449</td>
<td>69.987</td>
</tr>
<tr>
<td>4</td>
<td>90</td>
<td>2%</td>
<td>98.3</td>
<td>8.3</td>
<td>36.0</td>
<td>0.058</td>
<td>0.166</td>
<td>1.378</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>3%</td>
<td>98.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>2%</td>
<td>98.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>90%</td>
<td>98.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td>56.6</td>
<td>66.0</td>
<td>0.266</td>
<td>1.615</td>
<td>71.364</td>
</tr>
</tbody>
</table>

#### Portfolio (F+G)

<table>
<thead>
<tr>
<th>State</th>
<th>X</th>
<th>P(X)</th>
<th>E(X)</th>
<th>Losses</th>
<th>Weighted Tail</th>
<th>([E(X)-X]^{0.5} P(X))</th>
<th>([E(X)-X] P(X))</th>
<th>([E(X)-X]^2 P(X))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>150</td>
<td>3%</td>
<td>196.6</td>
<td>46.6</td>
<td>75.0</td>
<td>0.205</td>
<td>1.398</td>
<td>65.147</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>3%</td>
<td>196.6</td>
<td>46.6</td>
<td>75.0</td>
<td>0.205</td>
<td>1.398</td>
<td>65.147</td>
</tr>
<tr>
<td>2</td>
<td>190</td>
<td>2%</td>
<td>196.6</td>
<td>6.6</td>
<td></td>
<td>0.051</td>
<td>0.132</td>
<td>0.871</td>
</tr>
<tr>
<td>4</td>
<td>190</td>
<td>2%</td>
<td>196.6</td>
<td>6.6</td>
<td></td>
<td>0.051</td>
<td>0.132</td>
<td>0.871</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
<td>90%</td>
<td>196.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td>106.4</td>
<td>150.0</td>
<td>0.512</td>
<td>3.060</td>
<td>132.036</td>
</tr>
</tbody>
</table>

#### Risk Measures

<table>
<thead>
<tr>
<th>Risk measure</th>
<th>Investment F</th>
<th>Investment G</th>
<th>Sum F, G</th>
<th>Portfolio</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR 95%</td>
<td>8.300</td>
<td>8.300</td>
<td>16.600</td>
<td>46.600</td>
<td><strong>Fails subadditivity</strong></td>
</tr>
<tr>
<td>ES 95%</td>
<td>32.300</td>
<td>32.300</td>
<td>64.600</td>
<td>46.600</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM$_{0}$</td>
<td>0.050</td>
<td>0.050</td>
<td>0.100</td>
<td>0.010</td>
<td>Weakly subadditive</td>
</tr>
<tr>
<td>LPM$_{0.5}$</td>
<td>0.266</td>
<td>0.266</td>
<td>0.532</td>
<td>0.512</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM$_{1}$</td>
<td>1.615</td>
<td>1.615</td>
<td>3.230</td>
<td>3.060</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM$_{2}$</td>
<td>71.364</td>
<td>71.364</td>
<td>142.728</td>
<td>132.026</td>
<td>Subadditive</td>
</tr>
</tbody>
</table>

38
Finally, consider the case of Table 3, which shows risk measures for two individual loans, R and S, and the portfolio that combines these loans. These loans have a face value of $100, but different expected values, and offer diversification benefits in the domain of losses. In the first state of nature, loan R has a value of $50 while loan S has a value of $90. In the second state, loan R has a value of $90 while loan S has a value $50. In the third, fourth and fifth states of nature, both loans have the identical values of $96, $98 and $100 respectively.

In the case of loans R and S, the only measure that fails subadditivity is LPM$_2$. However, it is worth noting that VaR, ES and LPM$_1$ are all weakly subadditive and hence fail to reward the diversified portfolio through the risk measure. In this particular scenario, it is observed that the VaR measure is subadditive because diversification across the individual loans occurs outside the 95% confidence level upon which the VaR is based. That is, diversification occurs only in the first and second states of nature, which cover the 4% of the cumulative losses of the loans, starting from the largest loss. In a similar vein, ES fails to reward portfolio diversification (by assigning a lower risk value to the diversified portfolio than for the sum of the individual risk values) because diversification occurs outside the 95% confidence threshold. This serves as a reminder of the target dependence of risk measures based on a predetermined confidence level.
## Table 3
### Subadditivity for Loans R and S: Risk Measures

### Loan R

| State | X  | P(X) | E(X) | Losses | Weighted Tail | \(|E(X)-X|^1/2 P(X)| | \(|E(X)-X| P(X)| | \(|E(X)-X|^2 P(X)| |
|-------|----|------|------|--------|---------------|-----------------|-----------------|-----------------|
| 1     | 50 | 3%   | 98.24| 48.24  | 25.0          | 0.208           | 1.447           | 69.813          |
| 2     | 90 | 1%   | 98.24| 8.24   | 15.3          | 0.029           | 0.082           | 0.679           |
| 3     | 96 | 2%   | 98.24| 2.24   | 31.7          | 0.030           | 0.045           | 0.100           |
| 4     | 98 | 4%   | 98.24| 0.24   |               | 0.020           | 0.010           | 0.002           |
| 5     | 100| 90%  | 98.24|        |               |                 |                 |                 |

| Sum   | 100%|      | 72.0 | 0.287  | 1.584         | 70.595          |

### Loan S

| State | X  | P(X) | E(X) | Losses | Weighted Tail | \(|E(X)-X|^1/2 P(X)| | \(|E(X)-X| P(X)| | \(|E(X)-X|^2 P(X)| |
|-------|----|------|------|--------|---------------|-----------------|-----------------|-----------------|
| 1     | 90 | 3%   | 99.04| 9.04   | 45.0          | 0.090           | 0.271           | 2.452           |
| 2     | 50 | 1%   | 99.04| 49.04  | 8.3          | 0.070           | 0.490           | 24.049          |
| 3     | 96 | 2%   | 99.04| 3.04   | 32.0         | 0.035           | 0.061           | 0.185           |
| 4     | 98 | 4%   | 99.04| 1.04   |             | 0.041           | 0.042           | 0.043           |
| 5     | 100| 90%  | 99.04|        |             |                 |                 |                 |

| Sum   | 100%|      | 85.3 | 0.236  | 0.864         | 26.729          |

### Portfolio (R+S)

| State | X  | P(X) | E(X) | Losses | Weighted Tail | \(|E(X)-X|^1/2 P(X)| | \(|E(X)-X| P(X)| | \(|E(X)-X|^2 P(X)| |
|-------|----|------|------|--------|---------------|-----------------|-----------------|-----------------|
| 1     | 140| 3%   | 197.28| 57.28 | 70.0          | 0.227           | 1.718           | 98.430          |
| 2     | 140| 1%   | 197.28| 57.28 | 23.3          | 0.076           | 0.573           | 32.810          |
| 3     | 192| 2%   | 197.28| 5.28  | 64.0         | 0.046           | 0.106           | 0.558           |
| 4     | 196| 4%   | 197.28| 1.28  |             | 0.045           | 0.051           | 0.066           |
| 5     | 200| 90%  | 197.28|      |             |                 |                 |                 |

| Sum   | 100%|      | 157.3| 0.394  | 2.448         | 131.863         |

### Risk Measures

<table>
<thead>
<tr>
<th>Risk measure</th>
<th>Investment R</th>
<th>Investment S</th>
<th>Sum R, S</th>
<th>Portfolio</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR 95%</td>
<td>2.240</td>
<td>3.040</td>
<td>5.280</td>
<td>5.280</td>
<td>Weakly subadditive</td>
</tr>
<tr>
<td>LPM$_0$</td>
<td>0.100</td>
<td>0.100</td>
<td>0.200</td>
<td>0.100</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM$_{0.5}$</td>
<td>0.287</td>
<td>0.236</td>
<td>0.523</td>
<td>0.394</td>
<td>Subadditive</td>
</tr>
<tr>
<td>LPM$_1$</td>
<td>1.584</td>
<td>0.864</td>
<td>2.448</td>
<td>2.448</td>
<td>Weakly subadditive</td>
</tr>
<tr>
<td>LPM$_2$</td>
<td>70.595</td>
<td>26.729</td>
<td>97.324</td>
<td>131.863</td>
<td>Fails subadditivity</td>
</tr>
</tbody>
</table>
Let us consider the general findings regarding the subadditivity of the risk measures studied in this chapter.

To prove that a risk measure is not subadditive we need only provide a counter example for each risk measure where the subadditivity condition fails. The preceding examples show that VaR and LPM$_2$ fail the subadditivity condition. For both these measures, we have shown that portfolio diversification may increase the VaR for the portfolio, when the true position is one of lower risk for the portfolio.\textsuperscript{37}

The risk measures that conform to the subadditivity condition in our examples are ES and LPM (order $0 \leq n \leq 1$). Artzner, et al (1997, 1999) provide proofs that ES is subadditive, and Acerbi, Nordio and Sirtori (2001) extend the work of Artzner et al (1997, 1999) to show that ES is subadditive in cases where the underlying profit/loss distributions are discontinuous. We have identified cases where ES is weakly subadditive in the sense that it does not penalise diversification, but at the same time, the risk measure does not reward diversification by presenting a lower risk value for the diversified portfolio. This occurs when diversification in the distribution of returns occurs outside of the confidence level for the measurement of the shortfall. As discussed above, if the true underlying risk of the portfolio is less than the sum of the risks of the individual loans that make up the portfolio, then a risk measure should reflect this. Incentive-compatibility requires that managers are incentivised to seek out loans that are risk-reducing when combined in a portfolio, and to add them to the portfolio. If the portfolio risk measure does not reflect the lower risk, then the incentive to diversify is reduced.

Our results confirm the findings Barbosa and Ferreira (2004) that the LPM$_n$ categories of risk measures are subadditive when the order is $0 \leq n \leq 1$. The result for LPM$_1$ is not surprising given this measure closely resembles ES, the only difference being ES is based on a $\alpha$ confidence level while LPM$_1$ is typically based on deviations below the expected value for the portfolio. Although ES and LPM$_1$ are coherent risk

measures, we have previously discarded them from the acceptable list of incentive-compatible risk measures because they embody risk-neutrality in losses and thus fail to rank portfolios consistently in terms of stochastic dominance principles, and in particular, TSD. Similarly, we have shown that \( \text{LPM}_n \) of order \( 0 \leq n < 1 \) embodies a risk-seeking attitude on the part of investors, which again does not conform to the risk-preference function of the centre of the bank. Thus the subadditivity of the LPM risk measure within these specifications is not of consequence.

Table 4 summarises our results on the coherence of the five risk measure candidates selected for this paper.

<table>
<thead>
<tr>
<th>1. Shortfall probability</th>
<th>Translation Invariance</th>
<th>Monotonicity</th>
<th>Positive Homogeneity</th>
<th>Subadditivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. VaR</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>3. Expected Shortfall</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>4. LPM(_1)</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>5. LPM(_2)</td>
<td>No</td>
<td>Partial</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

6. CONCLUSION

Shortfall probability, VaR, expected shortfall and LPM\(_1\) are not incentive-compatible risk measures given the risk preferences of the centre of the banking firm. The use of these measures in the dominator of a RAPM may induce managers to select portfolios that are dominated in terms of TSD criteria. VaR, expected shortfall and LPM\(_1\) also display target dependence, meaning changes in the target loss threshold for these measures can impact on the risk-ordering of portfolios. The LPM\(_2\) risk measure is compatible with TSD and provides a risk-ordering of portfolios that is incentive-compatible. The measure also provides a consistent risk-ordering independent of the target threshold.
Although LPM\textsubscript{2} is consistent with stochastic dominance principles, it is not a coherent risk measure. LPM\textsubscript{2} fails each of the four axioms of a coherent risk measure. The failure of LPM\textsubscript{2} to meet the axioms of translation invariance and monotonicity is not of consequence in terms of goal congruence between the centre and managers. The failure of LPM\textsubscript{2} to meet the axiom of positive homogeneity is not of significance if performance is measured and remunerated on an individual loan basis, but is relevant if managers are measured and remunerated on a portfolio basis.\textsuperscript{38} This is because the LPM\textsubscript{2} for two identical loans that are combined in a portfolio will always exceed the sum of the LPM\textsubscript{2} of the individual loans. This may lead managers to reject loans that are valuable to the bank. The failure of LPM\textsubscript{2} to meet the axiom of subadditivity is relevant because the measure does not reward portfolio diversification and thus does not encourage managers to seek-out and add loans to their portfolios that provide risk-reducing benefits. This is also against the best interests of the centre of the bank.

If regulators or ratings agencies deem that total bank capital should be measured in terms of a bank solvency standard, which in turn is based on the probability of the bank defaulting on its debt, then the internal risk measure must diverge from the external measure of risk, where the objective of the internal measure is to achieve a disciplined and consistent analysis of risk based on the entire distribution of potential outcomes. It may not be desirable or possible for a single risk measure to meet competing objectives.

\textsuperscript{38} This is where loans with different distribution of returns are combined in a portfolio. In our setting, for example, this would be the case if loan portfolios A and B (or any other combination) were combined to create a third portfolio.
REFERENCES


